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From boundary to bulk in logarithmic CFT

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Abstract

The analogue of the charge-conjugation modular invariant for rational logarithmic conformal field theories is constructed. This is done by reconstructing the bulk spectrum from a simple boundary condition (the analogue of the Cardy ‘identity brane’). We apply the general method to the $c_{1,p}$ triplet models and reproduce the previously known bulk theory for $p = 2$ at $c = -2$. For general p we verify that the resulting partition functions are modular invariant. We also construct the complete set of $2p$ boundary states, and confirm that the identity brane from which we started indeed exists. As a by-product we obtain a logarithmic version of the Verlinde formula for the $c_{1,p}$ triplet models.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In the last few years logarithmic conformal field theories have increasingly attracted attention. They appear in various models of statistical physics, for example in the theory of (multi)critical polymers [1–3], percolation [4–6] and various critical (disordered) models [7–16]. During the last year lattice realizations of logarithmic conformal field theories have also been found [17–19]. In a separate development WZW models on supergroups (that also exhibit logarithmic behaviour) have been more intensively studied [20–24]. These supergroup theories are likely to play an important role for the world-sheet description of string theory on AdS spaces. Finally, logarithmic conformal field theories are also interesting from an abstract point of view since they fall outside the well-studied class of rational conformal field theories and thus represent a first step towards understanding at least some aspects of non-rational theories. The abstract structure of logarithmic conformal field theories has also been studied, starting from [25, 26], and more recently in the mathematical literature [27–33]. Reviews about different aspects of logarithmic conformal field theories are [34–36].

While there has recently been some interesting progress with the supergroup theories, the best understood logarithmic conformal field theory continues to be the rational triplet theory at $c = -2$ [37]. It is the only logarithmic conformal field theory for which all structures have been understood in detail from a conformal field theory point of view. In particular, the fusion rules of this theory were derived from first principles in [38], and a consistent local theory, whose amplitudes satisfy crossing symmetry, has been constructed by solving the conformal bootstrap in [39]. More recently, a careful analysis of the boundary theory has been performed in [40], and a consistent set of boundary states has been found. (Some of these results were anticipated in [41], see also [42–45] for earlier discussions.)

The local bulk theory of the triplet theory [39] is actually quite complicated—its only simple description is in terms of the symplectic fermions [3, 46] that are quite special for the $c = -2$ triplet theory—but the boundary theory of [40] turned out to be remarkably simple. The boundary states are labelled by the irreducible representations of the triplet theory, while the open string spectrum consists precisely of the representations that appear in the fusion of these irreducible representations. (These representations involve then in general indecomposable logarithmic representations.) In this paper, we show how the bulk theory can actually be obtained in a very natural manner from the boundary theory. Our analysis applies to all rational logarithmic conformal field theories; for the case of the $c_{1,p}$ models we can describe the resulting bulk theory very explicitly, and for $p = 2$ it coincides with the original $c = -2$ bulk theory of [39].

Our method is based on insights into the general structure of conventional (non-logarithmic) rational conformal field theories that have been obtained during the last few years [47–50]. In particular, it has become clear that a good way to describe a given rational conformal field theory is by starting from its boundary theory: given the spectrum of boundary fields on a single boundary condition (that preserves the full chiral algebra) as well as the associative operator product of these fields, one can reconstruct the bulk theory for which this boundary theory describes an allowed boundary condition. In particular, this allows one to solve the complicated conformal bootstrap equations in terms of the much simpler problem of constructing an associative operator product on the boundary.

The basic idea behind this reconstruction can be described as follows. One can argue on general grounds that the disc correlation functions that involve one bulk field and one boundary field are non-degenerate in the bulk field insertion. This is to say, for any non-trivial bulk field there exists a boundary field such that the corresponding disc correlator does not vanish. Knowing the boundary spectrum thus gives constraints on the possible size of the bulk theory. Furthermore, the correlation functions involving one bulk field and two boundary fields must essentially be independent of the order in which the latter appear on the boundary since one can take one of them around the circle (see figure 2). It was shown in [47, 50] that in the non-logarithmic rational case the bulk theory is then simply the largest possible representation of the two chiral algebras that satisfies these two constraints. Furthermore one can determine from these data also the bulk correlation functions, etc.

While the corresponding statement is not yet known for the logarithmic case (that falls outside the mathematical analysis of [47, 50]), it is clear that these two conditions also have to hold in the general (logarithmic) situation. We can therefore use these ideas to constrain the possible spectrum of the bulk theory starting from a boundary condition. Given the above observations about the boundary theory of the $c = -2$ triplet model, it seems very likely that at least all $c_{1,p}$ triplet models will have a boundary condition whose boundary spectrum consists just of the vacuum representation of the triplet algebra. (This is the boundary condition associated with the irreducible vacuum representation.) Starting from such a boundary condition we can analyse the above constraints and construct the largest

space that is compatible with them. For the specific case of the $c_{1,p}$ triplet models (for which some aspects of the allowed representations are known) we can then be even more specific and describe the resulting bulk space very explicitly (see (4.22)). As we show in detail, it leads to a modular invariant partition function and supports boundary conditions in one-to-one correspondence with the irreducible representations of the triplet algebra. This gives strong support to the assertion that this describes in fact the correct bulk theory. It also reduces to the known bulk theory [39] for $c = -2$ and is compatible with the predictions (based on the analogy with supergroups) of [24].

In the usual rational case, the bulk theory corresponding to the boundary condition whose spectrum consists just of the vacuum representation itself, is the charge-conjugation modular invariant [51, 52]. The above theories should therefore be thought of as the analogue of the charge-conjugation construction. One may suspect that there will also be other consistent bulk theories (with other modular invariant partition functions [29, 34, 53]). It would be interesting to study this for the example of the triplet theories.

As a by-product of our analysis we find an expression for the boundary states of the $c_{1,p}$ models in terms of the S -matrices. Since the open string multiplicities are determined in terms of the fusion rules, this then leads to a Verlinde-like formula for the fusion rules of these models. (More precisely, the formula describes the product in the associated Grothendieck ring.) Given the abstract form of the formula it is very natural to suspect that it will generalize to other logarithmic rational conformal field theories.

The paper is organized as follows. In section 2, we explain the general method of reconstructing the bulk theory from a given boundary condition. In section 3, we concentrate on the case that the boundary only has the vacuum representation in its boundary spectrum and derive the constraints on the possible bulk space in the general logarithmic case. We also give a fairly explicit description of the largest such space. In section 4, these ideas are then applied to the $c_{1,p}$ triplet models. In particular, we give a detailed description of the bulk spectrum for general p in section 4.3, and show that it reproduces the known result for $p = 2$. We also show in section 4.5 that it leads to a modular invariant partition function. Finally, in section 5 we analyse the Cardy condition for this bulk spectrum, and show how to construct boundary states in one-to-one correspondence with the irreducible representations. We also discuss the Verlinde formula for logarithmic rational conformal field theories there. Section 6 contains our conclusions. There are a number of appendices in which some of the more technical material is described. Among other things, we also conjecture there the fusion rules for the general \mathcal{W}_p triplet algebras at $c = c_{1,p}$ (see (C.4), (C.6) and (C.8)).

2. Constructing the space of bulk states

In this section, we will generalize one key element of non-logarithmic rational conformal field theories to the logarithmic case, namely the construction of the space of bulk fields from a given algebra of boundary fields [47].

2.1. The bulk space from disc amplitudes

Suppose we are given a conformal field theory (logarithmic or not) that is defined on surfaces with (and without) boundaries. In particular the theory is defined on the unit disc, where at the boundary of the disc we have chosen one of the possible boundary conditions (that we shall denote by γ). Consider now the correlator involving an arbitrary bulk field in the interior of the disc, together with a single boundary field on the boundary. By the usual $SU(1, 1)$ symmetry of disc correlation functions, we may assume without loss of generality that the

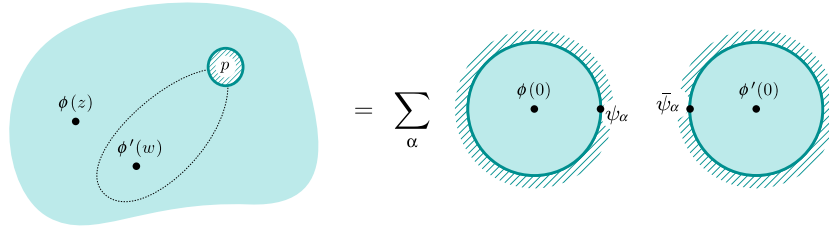


Figure 1. The correlator of two bulk fields on the complex plane with a little hole can be written as a sum of products of disk correlators by factorizing along the dashed interval. (All factors arising from the conformal transformations to the disc have been absorbed into the bases ψ_α and $\bar{\psi}_\alpha$.)

bulk field is inserted at $z = 0$, while the boundary field sits at $z = 1$. This correlator defines a bilinear pairing

$$b : \mathcal{H}_{\text{bulk}} \times \mathcal{H}_{\text{bnd}} \rightarrow \mathbb{C}, \quad b(\phi, \psi) = \langle \phi(0)\psi(1) \rangle_\gamma, \tag{2.1}$$

where $\mathcal{H}_{\text{bulk}}$ is the space of bulk fields, while \mathcal{H}_{bnd} denotes the space of boundary fields on the boundary γ .

We will now argue that this pairing is non-degenerate in the first argument. This means that for any nonzero bulk field ϕ , there exists a boundary field ψ such that the correlator does not vanish, $b(\phi, \psi) \neq 0$. To see this we recall that, by definition, the two-point functions on the sphere define a non-degenerate bilinear pairing on the space of bulk fields. (This is to say, if a bulk field ϕ vanishes in all two-point functions on the sphere, then we have in fact $\phi = 0$.) This property should not change if we consider instead the two-point function on the sphere with a little boundary circle around some point p far away from the insertion points of the bulk fields. But then we can use factorization along an interval starting and ending on this boundary circle to express the correlation function as a sum over products of disc correlators (see figure 1). It is then clear that the bulk-boundary correlators must be non-degenerate in the bulk fields in order for the above two-point function to be non-degenerate. This proves that the bilinear pairing b is non-degenerate with respect to the first argument. We note in passing that the argument does not imply that b must be non-degenerate in the boundary fields as well; in fact, this is not true in general. (Consider for example a superposition of boundary conditions and take ψ to be a boundary changing operator. Then $b(\phi, \psi) = 0$ for all bulk fields ϕ .)

If we are given a boundary condition γ with some space of boundary fields \mathcal{H}_{bnd} , the condition that the bulk-boundary correlation functions must be non-degenerate in the bulk fields will give restrictive constraints on the structure of the bulk space $\mathcal{H}_{\text{bulk}}$. This will in particular be the case if \mathcal{H}_{bnd} is rather small, for example if it just consists of the chiral algebra \mathcal{V} of the theory itself. One can then turn the logic around and ‘reconstruct’ the bulk space from the boundary condition. This is what we shall be doing in the following. First, however, we briefly want to elaborate on the general situation.

2.2. Constraints on the bulk space from a generic brane

We denote the chiral algebra of the bulk theory (i.e. the conformal vertex algebra of the holomorphic degrees of freedom) by \mathcal{V} , and by $\mathcal{V} \times \bar{\mathcal{V}}$ the holomorphic and anti-holomorphic copy of \mathcal{V} in the bulk. We shall always consider boundary conditions that preserve \mathcal{V} ; thus we assume that for any holomorphic field W of \mathcal{V} we have

$$W(z) = \bar{W}(\bar{z}), \quad z = \bar{z}, \tag{2.2}$$

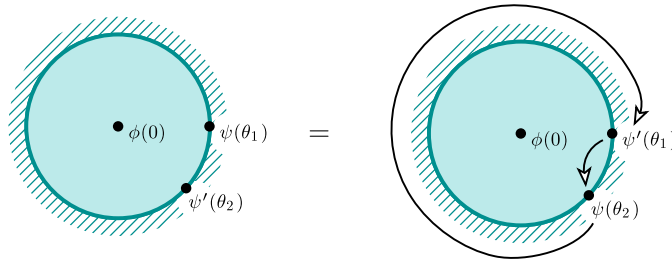


Figure 2. A constraint on the possible bulk fields ϕ : Inserting two boundary fields in reversed order is equivalent to analytic continuation around the disk. The two disk correlators are fixed separately by b and the operator product expansion on \mathcal{H}_{bnd} .

where \bar{W} is the corresponding field in $\bar{\mathcal{V}}$. (We have written this condition for the case where instead of the disk we are considering the upper half plane with boundary the real axis.) The space of boundary fields \mathcal{H}_{bnd} is then a representation of \mathcal{V} ; as usual, the operator product on \mathcal{H}_{bnd} must be associative.

The arguments of the previous subsection now imply that the space of bulk fields $\mathcal{H}_{\text{bulk}}$ must have the property that:

- (1) There exists a pairing $b : \mathcal{H}_{\text{bulk}} \times \mathcal{H}_{\text{bnd}} \rightarrow \mathbb{C}$ compatible with the action of \mathcal{V} and non-degenerate in the first argument.

The compatibility condition with the \mathcal{V} -action follows from the usual contour deformation arguments involving the holomorphic fields in \mathcal{V} ; it will be given in more detail in section 3.1 and appendix A. Using the associativity of the operator product expansion on the boundary, the pairing b then also determines uniquely the disk correlator of an arbitrary number of boundary fields with one bulk field. A second constraint is then:

- (2) A disk correlator with bulk insertion $\phi(0)$ and boundary insertions $\psi(\theta_1)\psi'(\theta_2)$ has to be related to the correlator with reversed boundary insertions $\psi'(\theta_1)\psi(\theta_2)$ by analytic continuation (see figure 2).

This second condition is just one of the sewing constraints for conformal field theories with boundary [54, figure 9(d)]. For rational conformal field theories, using the language of [47, section 5.3], it amounts to the statement that bulk fields are in the image of a certain projector, while in the approach of [55] it is definition 5.11.

For non-logarithmic rational conformal field theories one can show that $\mathcal{H}_{\text{bulk}}$ is uniquely determined by the boundary condition γ (i.e. by the associative algebra of boundary fields on γ), and that it is simply the largest $\mathcal{V} \times \bar{\mathcal{V}}$ -representation that satisfies these constraints (see [47, lemma 5.6] and [50]). While we do not yet know how to prove the corresponding statement in the general logarithmic case, it is clear that any consistent $\mathcal{H}_{\text{bulk}}$ must satisfy at least these constraints. Furthermore the examples we shall study below suggest that $\mathcal{H}_{\text{bulk}}$ is again (also in the logarithmic case) simply the largest $\mathcal{V} \times \bar{\mathcal{V}}$ -representation that satisfies (1) and (2).

3. The identity brane

We now want to discuss the construction of the bulk space for the simplest case where the boundary spectrum consists just of the vacuum representation of the chiral algebra, i.e. for

which $\mathcal{H}_{\text{bnd}} = \mathcal{V}$. In the non-logarithmic rational case such a brane exists in the charge conjugation theory, namely as the Cardy brane associated with the vacuum representation [56]. For the logarithmic triplet model at $c = -2$ for which the boundary conditions were analysed in detail in [40], we also found one such brane.

In the following we shall thus assume that we have a boundary condition γ for which $\mathcal{H}_{\text{bnd}} = \mathcal{V}$. We want to construct a bulk theory $\mathcal{H}_{\text{bulk}}$ that satisfies conditions (1) and (2) relative to this boundary. Since $\mathcal{H}_{\text{bnd}} = \mathcal{V}$, condition (2) is simply implied by the fact that $\mathcal{H}_{\text{bulk}}$ is a representation of $\mathcal{V} \times \bar{\mathcal{V}}$. Thus we only need to find a solution to condition (1). To this end we start with some large space of potential bulk states $\hat{\mathcal{H}}$. We then calculate the correlation functions of bulk states in $\hat{\mathcal{H}}$ on the disc with the boundary condition γ ; these are determined by the chiral symmetry up to some coupling constants (normalizations of three-point blocks). For any choice of these coupling constants we then find the subspace $\mathcal{N} \subset \hat{\mathcal{H}}$ of potential bulk states that vanishes in all such disc correlation functions; for the given choice of coupling constants the actual bulk space is thus the quotient $\mathcal{H}_{\text{bulk}} = \hat{\mathcal{H}}/\mathcal{N}$. Obviously, the null-space \mathcal{N} (and therefore $\mathcal{H}_{\text{bulk}}$) depends on the choice of these coupling constants, but as we shall see, the resulting space is essentially independent of these choices as long as we pick generic values. We thus define the bulk space to be the largest such space as we vary the coupling constants. For the $c_{1,p}$ triplet models we will see in sections 4 and 5 that the resulting space leads to a modular invariant partition function and gives rise to the expected boundary states, in particular one with $\mathcal{H}_{\text{bnd}} = \mathcal{V}$. The fact that this last boundary condition satisfies the Cardy constraint is not *a priori* guaranteed, and hence provides a consistency check on our approach.

After this informal description of the strategy, we now want to give more details of the construction.

3.1. The universal property defining $\mathcal{H}_{\text{bulk}}$

As we have just mentioned we shall from now on assume that \mathcal{H}_{bnd} is just the chiral algebra itself $\mathcal{H}_{\text{bnd}} = \mathcal{V}$. Let us start with a simple ansatz for the space of potential bulk states $\hat{\mathcal{H}}$, namely that $\hat{\mathcal{H}}$ is the direct sum of tensor products of representations of \mathcal{V} and $\bar{\mathcal{V}}$; a particular term in this sum will thus be of the form $M \otimes \bar{N}$, where M and N are representations of \mathcal{V} . (The bar on N indicates that it describes the right-moving degrees of freedom that form a representation of $\bar{\mathcal{V}}$.) A disk correlator with one bulk insertion in $M \otimes \bar{N}$ at $z = 0$ and a boundary insertion at $z = 1$ can be mapped conformally to the upper half plane with a boundary insertion at 0 and a bulk insertion at i . Since the boundary condition preserves the chiral algebra (2.2) we can use the doubling trick [57] to write this correlator as the three-point block on the complex plane with an insertion of \mathcal{V} at 0, while M and N are inserted at $\pm i$. Every such conformal block β defines a multilinear map $M \times N \times \mathcal{V} \rightarrow \mathbb{C}$ that obeys invariance conditions with respect to the \mathcal{V} -actions which are listed explicitly in appendix A. Furthermore, the three-point block β gives rise to a bulk-boundary correlator, and thus to an associated pairing $b_\beta : (M \otimes \bar{N}) \times \mathcal{V} \rightarrow \mathbb{C}$, $(a \otimes b, v) \mapsto \beta(a, b, v)$.

Similarly, if $\hat{\mathcal{H}} = \bigoplus_k M_k \otimes \bar{N}_k$ then the pairings that are compatible with the \mathcal{V} -action are of the form $\sum_k b_{\beta_k}$, where β_k is a three-point block $M_k \times N_k \times \mathcal{V} \rightarrow \mathbb{C}$. In fact, the possible pairings can also be described for an arbitrary $\mathcal{V} \times \bar{\mathcal{V}}$ -representation $\hat{\mathcal{H}}$, not just one of the form $\bigoplus_k M_k \otimes \bar{N}_k$; as is shown in appendix A, they have to satisfy condition (A.3). We denote the space of all such pairings by $B(\hat{\mathcal{H}})$.

Since we are only interested in the non-degeneracy of b in the first (bulk) entry, it is convenient to associate with each $b \in B(\hat{\mathcal{H}})$ the map $\underline{b} : \hat{\mathcal{H}} \rightarrow \mathcal{V}^*$ given by $\underline{b}(\phi) = b(\phi, \cdot)$. Condition (1) of the previous section is then just the requirement that \underline{b} must be injective (i.e. that its kernel is trivial). In general, this will not be the case for our ansatz $\hat{\mathcal{H}}$, but it is easy to

rectify this problem. We denote the kernel of \underline{b} by $\mathcal{N} = \ker(\underline{b}) \subset \hat{\mathcal{H}}$. Then for the quotient space $\mathcal{H} = \hat{\mathcal{H}}/\mathcal{N}$ the induced pairing on $\mathcal{H} \times \mathcal{V}$ is by construction non-degenerate in the first argument. As is shown in appendix B, \mathcal{H} is still a representation of $\mathcal{V} \times \bar{\mathcal{V}}$ since \mathcal{N} is; this is for example necessary to guarantee that condition (2) continues to hold.

Obviously, the space \mathcal{H} we end up with depends to a certain extent on the choice of the three-point blocks β_k . (For example, we could take all $\beta_k = 0$, in which case \mathcal{H} would be the zero space.) We expect that the actual bulk space that contains the boundary condition in question is as large as it can be. In order to make this precise we need to define what we mean by a ‘maximal solution’. A $\mathcal{V} \times \bar{\mathcal{V}}$ representation \mathcal{H} together with a pairing $b \in B(\mathcal{H})$ is a maximal solution to (1) if and only if \underline{b} is injective and the following universal property holds: for any pair (\mathcal{H}_1, b_1) such that $b_1 \in B(\mathcal{H}_1)$ and $\underline{b}_1 : \mathcal{H}_1 \rightarrow \mathcal{V}^*$ is injective there exists a unique injective intertwiner $f : \mathcal{H}_1 \rightarrow \mathcal{H}$ such that the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{H} & \xrightarrow{\underline{b}} & \mathcal{V}^* \\
 \uparrow \exists! f & \nearrow \underline{b}_1 & \\
 \mathcal{H}_1 & &
 \end{array}
 \tag{3.1}$$

In fact, if an intertwiner f exists, it is automatically unique and injective because both \underline{b} and \underline{b}_1 are injective. Furthermore, a maximal solution to condition (1) is unique up to isomorphism.

3.2. The bulk space in terms of projective covers

What we have said so far is completely general, but in order to be more specific we need to assume some properties about the representations of the chiral algebra \mathcal{V} (more precisely, the logarithmic modules [27], or generalized modules [33, section 2]). We assume that

- (i) \mathcal{V} has only finitely many inequivalent irreducible representations.
- (ii) Each \mathcal{V} -representation M has a projective cover $P(M)$.

As will become clear in section 4, the triplet algebras \mathcal{W}_p for $p \geq 2$ satisfy these conditions. More abstractly, one may expect that the representation category of a rational logarithmic conformal field theory is described by a finite tensor category (see, e.g., [32, 58]); then these conditions are automatically satisfied. We shall also use the following properties of $\mathcal{V} \times \bar{\mathcal{V}}$ -representations³:

- (I) Every $\mathcal{V} \times \bar{\mathcal{V}}$ -representation X is isomorphic to a quotient of the tensor product (over \mathbb{C}) $M_X \otimes \bar{N}_X$ of two \mathcal{V} -representations M_X and N_X by a subrepresentation.
- (II) The space of intertwiners $\text{Hom}_{\mathcal{V} \times \bar{\mathcal{V}}}(M \otimes \bar{N}, M' \otimes \bar{N}')$ is isomorphic to the tensor product of intertwiner spaces $\text{Hom}_{\mathcal{V}}(M, M') \otimes \text{Hom}_{\mathcal{V}}(N, N')$.

With these preparations we now proceed as follows. As our starting point we take the space of potential bulk states to be

$$\hat{\mathcal{H}} = \bigoplus_{k \in \text{Irr}} P_k \otimes \bar{P}_k^*,
 \tag{3.2}$$

where Irr labels the finitely many irreducible representations U_i , $i \in \text{Irr}$ of \mathcal{V} . Here P_k is the projective cover of U_k , and P_k^* is the conjugate representation to P_k . (More precisely, P_k is

³ These should hold for reasonable chiral algebras \mathcal{V} . For example (II) follows if the intertwiner spaces $\text{Hom}_{\mathcal{V}}(M, N)$ are finite dimensional, and the generalized L_0 -eigenspaces of the \mathcal{V} -representations are finite dimensional.

the contragredient module to P_k —for a definition see, e.g., [33, section 2].) The bar over the second space in the tensor product indicates that these degrees of freedom refer to right movers.

We also need to make an ansatz for the pairing b , or equivalently for the three-point blocks β_k . In fact, there is an almost canonical choice we can make: the space of three-point blocks involving any \mathcal{V} -representation M , its dual representation M^* and \mathcal{V} contains a preferred element that we shall denote by ev_M . To define ev_M we use a conformal transformation to move the insertion points of the three-point block such that M^* is inserted at infinity, while \mathcal{V} is inserted at $z = 1$ and M at $z = 0$. The three-point block ev_M is then uniquely determined by the condition that upon inserting the vacuum vector Ω at $z = 1$, the resulting pairing $M^* \times M \rightarrow \mathbb{C}$ is just the canonical pairing of a vector space with its dual. (The invariance conditions of appendix A then determine ev_M for any other combination of states.)

For the case at hand $M = P_k$, and we can thus define the pairing b on $\hat{\mathcal{H}} \times \mathcal{V}$ to be given by

$$b_{ev} = b_\beta \quad \text{with} \quad \beta = \sum_{k \in \text{Irr}} ev_{P_k}. \quad (3.3)$$

The kernel \mathcal{N}_{ev} of b_{ev} is non-trivial in general, but as we shall see, the resulting quotient space

$$\mathcal{H}_{\text{bulk}} = \hat{\mathcal{H}} / \mathcal{N}_{ev} \quad (3.4)$$

will define a maximal solution to (1). Before we can prove this statement, we need to make a few observations about a certain class of three-point blocks.

3.3. Three-point blocks and the kernel of b_{ev}

Suppose we have a three-point block $\beta: M \times N \times \mathcal{V} \rightarrow \mathbb{C}$. Since one of the three representations (namely \mathcal{V}) is just the vacuum representation, every such three-point block defines a linear map $\beta^\sharp: N \rightarrow M^*$ that intertwines the action of \mathcal{V} , i.e. satisfies $\beta^\sharp \circ W_n = W_n \circ \beta^\sharp$ for every W_n in \mathcal{V} . It is then clear that we can write β as

$$\beta(m, n, v) = ev_M(m, \beta^\sharp(n), v), \quad (3.5)$$

where $m \in M, n \in N$ and $v \in \mathcal{V}$.

Similarly, if we have an intertwiner g of the chiral algebra mapping the \mathcal{V} -representations M to N , then the three-point blocks ev_M and ev_N are related as

$$ev_N(g(m), n^*, v) = ev_M(m, g^*(n^*), v), \quad (3.6)$$

where $g^* \in \text{Hom}_{\mathcal{V}}(N^*, M^*)$ is the linear map dual to g . Here $m \in M, n^* \in N^*$ and $v \in \mathcal{V}$. Equation (3.6) can be verified by taking the insertion points to 0, 1 and ∞ , and noting that g commutes in particular with the modes of the Virasoro algebra.

We are now in a position to give a good description of the kernel of b_{ev} , $\mathcal{N}_{ev} = \ker(b_{ev})$. We want to describe it as the span of the images of intertwiners $g: P_k \otimes \bar{P}_l^* \rightarrow \hat{\mathcal{H}}$; this is always possible since by (I) any $\mathcal{V} \times \bar{\mathcal{V}}$ -representation can be written as a quotient space of a suitable tensor product which—by passing to projective covers—we may decompose into a direct sum of tensor products of indecomposable projective representations. The image of such a map lies in \mathcal{N}_{ev} if and only if $b_{ev} \circ g = 0$. Thus we can write

$$\mathcal{N}_{ev} = \ker(b_{ev}) = \text{span}_{\mathbb{C}}\{\text{im}(g) \mid g: P_k \otimes \bar{P}_l^* \rightarrow \hat{\mathcal{H}} \text{ with } b_{ev} \circ g = 0, k, l \in \text{Irr}\}. \quad (3.7)$$

To characterise the relevant intertwiners g , we denote by H_{kl} the vector space

$$H_{kl} = \bigoplus_{i \in \text{Irr}} \text{Hom}_{\mathcal{V}}(P_k, P_i) \otimes \text{Hom}_{\mathcal{V}}(P_i, P_l), \quad (3.8)$$

where $k, l \in \text{Irr}$ and $\text{Hom}_{\mathcal{V}}(P_r, P_s)$ is the space of intertwiners from P_r to P_s . On H_{kl} we define two maps as follows. First, we have the composition map $c_{kl} : H_{kl} \rightarrow \text{Hom}_{\mathcal{V}}(P_k, P_l)$ which acts on each component $f_i \otimes g_i$ by composition

$$c_{kl}(f_i \otimes g_i) = g_i \circ f_i \in \text{Hom}_{\mathcal{V}}(P_k, P_l). \quad (3.9)$$

In addition we have the map $d_{kl} : H_{kl} \rightarrow \text{Hom}_{\mathcal{V} \times \bar{\mathcal{V}}}(P_k \otimes \bar{P}_l^*, \hat{\mathcal{H}})$ defined by setting

$$d_{kl}(f_i \otimes g_i) = f_i \otimes \bar{g}_i^* \in \text{Hom}_{\mathcal{V} \times \bar{\mathcal{V}}}(P_k \otimes \bar{P}_l^*, \hat{\mathcal{H}}), \quad (3.10)$$

where $\bar{g}_i^* \in \text{Hom}_{\mathcal{V}}(P_l^*, P_i^*)$ is the dual map to $g_i \in \text{Hom}_{\mathcal{V}}(P_i, P_l)$ (and the bar indicates again that \bar{g}_i^* acts now on the right movers). Note that d_{kl} is an isomorphism of vector spaces, as follows from (II) above. The key observation is now that (3.6) implies

$$\begin{aligned} \text{ev}_{P_i}([c_{kl}(f_i \otimes g_i)](w_k), \bar{w}_l, v) &\equiv \text{ev}_{P_i}([g_i \circ f_i](w_k), \bar{w}_l, v) \\ &= \text{ev}_{P_i}(f_i(w_k), \bar{g}_i^*(\bar{w}_l), v) \equiv b_{\text{ev}}([d_{kl}(f_i \otimes g_i)](w_k \otimes \bar{w}_l), v), \end{aligned} \quad (3.11)$$

where $w_k \in P_k$, $\bar{w}_l \in \bar{P}_l^*$ and $v \in \mathcal{V}$ are arbitrary. It therefore follows that

$$\underline{b}_{\text{ev}} \circ d_{kl}(F) = 0 \quad \text{if and only if} \quad F \in \ker(c_{kl}). \quad (3.12)$$

Since d_{kl} is an isomorphism, every map g in (3.7) can be written as $d_{kl}(F)$ for an appropriate $F \in H_{kl}$. Hence the expression (3.7) for the kernel of $\underline{b}_{\text{ev}}$ can be rewritten as

$$\mathcal{N}_{\text{ev}} = \text{span}_{\mathbb{C}}\{\text{im}(d_{kl}(F)) \mid F \in \ker(c_{kl}), k, l \in \text{Irr}\}. \quad (3.13)$$

The space of bulk states is then defined as in (3.4). We also denote by b_{disc} the pairing on $\mathcal{H}_{\text{bulk}} \times \mathcal{V}$ induced by b_{ev} . It is shown in section 3.5 below that $(\mathcal{H}_{\text{bulk}}, b_{\text{disc}})$ is in fact maximal.

This completes our construction of the bulk space corresponding to the identity brane. For non-logarithmic rational conformal field theories we have $P_i = U_i$ and one easily verifies that the linear maps c_{kl} all have trivial kernel. Thus one recovers the space of bulk states of the charge-conjugation modular invariant theory, $\mathcal{H}_{\text{bulk}} = \bigoplus_{k \in \text{Irr}} U_k \otimes \bar{U}_k^*$. We shall show in section 4.4 that for the case of the $c = -2$ triplet theory, the above construction reproduces the known bulk spectrum [39]. We shall also see that it leads to a very natural bulk spectrum for the other $c_{1,p}$ triplet models that is in particular modular invariant.

Before turning to the proof that $(\mathcal{H}_{\text{bulk}}, b_{\text{disc}})$ is in fact maximal we want to show that it defines at least a local theory.

3.4. Locality

Locality of the bulk theory requires that the operator $\exp(2\pi i(L_0 - \bar{L}_0))$ acts as the identity on $\mathcal{H}_{\text{bulk}}$. We want to show now that this is requirement is automatically satisfied by the above construction.

First, we note that $e^{2\pi i L_0}$ commutes with all generators of \mathcal{V} and that it therefore defines an intertwiner from any \mathcal{V} -representation to itself. Consider now the element

$$t = e^{2\pi i L_0} \otimes \text{id} - \text{id} \otimes e^{2\pi i L_0} \in \text{Hom}_{\mathcal{V}}(P_k, P_k) \otimes \text{Hom}_{\mathcal{V}}(P_k, P_k) \subset H_{kk}. \quad (3.14)$$

It is obvious that $c_{kk}(t) = 0$ and hence t lies in the kernel of c_{kk} . It then follows from (3.13) that the image of $P_k \otimes \bar{P}_k^*$ under $e^{2\pi i L_0} \otimes \text{id} - \text{id} \otimes e^{2\pi i L_0}$ lies in \mathcal{N}_{ev} , for all $k \in \text{Irr}$. In the quotient space $\mathcal{H}_{\text{bulk}}$ we therefore have $e^{2\pi i L_0} = e^{2\pi i \bar{L}_0}$, which yields the desired answer upon acting on both sides with $e^{-2\pi i \bar{L}_0}$.

The above argument implies in particular that the torus partition function for $\mathcal{H}_{\text{bulk}}$ is invariant under $\tau \mapsto \tau + 1$. We do not have a general proof that it is also invariant under $\tau \mapsto -1/\tau$, but we shall be able to show the full modular invariance for the $c_{1,p}$ triplet theories (see section 4.5).

3.5. Proof of maximality

We will now prove that the pair $(\mathcal{H}_{\text{bulk}}, b_{\text{disc}})$ has the universal property (3.1). Let (\mathcal{H}_1, b_1) be any pair such that $b_1 \in B(\mathcal{H}_1)$ and \underline{b}_1 is injective. Because of (I) above, there exist projective representations P and Q , as well as a subrepresentation K of $P \otimes \bar{Q}$ such that $\mathcal{H}_1 \cong (P \otimes \bar{Q})/K$. Let $\pi_1 : P \otimes \bar{Q} \rightarrow \mathcal{H}_1$ be the corresponding projection, and set $\underline{d} = \underline{b}_1 \circ \pi_1$. Since \underline{b}_1 is injective, we have $\ker(\underline{d}) = K$.

Next we write P in terms of indecomposable projectives as $P = \bigoplus_{i \in \text{Irr}} n_i P_i$ and denote by $\iota_k^\mu : P_k \rightarrow P$, $\mu = 1, \dots, n_k$, the embedding of P_k into the μ th copy of P_k in P . Similarly, $r_k^\mu : P \rightarrow P_k$ is the projection onto the μ th copy of P_k . Then

$$r_k^\mu \circ \iota_l^\nu = \delta_{k,l} \delta_{\mu,\nu} \text{id}_{P_k} \quad \text{and} \quad \sum_{k \in \text{Irr}} \sum_{\mu=1}^{n_k} \iota_k^\mu \circ r_k^\mu = \text{id}_P. \quad (3.15)$$

Let β be the conformal three-point block $P \times Q \times \mathcal{V} \rightarrow \mathbb{C}$ such that $d = b_\beta$. According to (3.5) we can write $\beta(p, q, v) = \text{ev}_P(p, \beta^\sharp(q), v)$ with β^\sharp an intertwiner from Q to P^* . This in turn implies that $\underline{d} = \underline{b}_{\text{ev}_P} \circ (\text{id}_P \otimes \bar{\beta}^\sharp)$, where (as usual) the bar over β^\sharp indicates that it now acts on the right movers. Thus we can define an intertwiner $\hat{\varphi} : P \otimes \bar{Q} \rightarrow \hat{\mathcal{H}}$ (with $\hat{\mathcal{H}}$ as given in (3.2)) by

$$\hat{\varphi} = \sum_{k \in \text{Irr}} \sum_{\mu=1}^{n_k} (r_k^\mu \otimes (\bar{r}_k^\mu)^*) \circ (\text{id}_P \otimes \bar{\beta}^\sharp). \quad (3.16)$$

This intertwiner obeys

$$\begin{aligned} \underline{b}_{\text{ev}} \circ \hat{\varphi} &= \sum_{k,\mu} \underline{b}_{\text{ev}_{P_k}} \circ (r_k^\mu \otimes \bar{r}_k^{\mu*}) \circ (\text{id}_P \otimes \bar{\beta}^\sharp) = \sum_{k,\mu} \underline{b}_{\text{ev}_P} \circ (\text{id}_P \otimes (\bar{r}_k^{\mu*} \circ r_k^\mu)) \circ (\text{id}_P \otimes \bar{\beta}^\sharp) \\ &= \sum_{k,\mu} \underline{b}_{\text{ev}_P} \circ (\text{id}_P \otimes (\bar{r}_k^\mu \circ r_k^\mu)^*) \circ (\text{id}_P \otimes \bar{\beta}^\sharp) = \underline{b}_{\text{ev}_P} \circ (\text{id}_P \otimes \bar{\beta}^\sharp) = \underline{d}, \end{aligned} \quad (3.17)$$

where we used (3.6) and (3.15). Let φ be the map from $P \otimes \bar{Q}$ to the quotient $\mathcal{H}_{\text{bulk}} = \hat{\mathcal{H}}/\mathcal{N}_{\text{ev}}$ induced by $\hat{\varphi}$, i.e. $\varphi = \pi \circ \hat{\varphi}$, where $\pi : \hat{\mathcal{H}} \rightarrow \mathcal{H}_{\text{bulk}}$ is the projection to the quotient. It then follows that for $x \in P \otimes \bar{Q}$,

$$\underline{d}(x) = 0 \quad \Rightarrow \quad \underline{b}_{\text{ev}} \circ \hat{\varphi}(x) = 0 \quad \Rightarrow \quad \hat{\varphi}(x) \in \ker(\underline{b}_{\text{ev}}) \quad \Rightarrow \quad \varphi(x) = 0. \quad (3.18)$$

Thus φ vanishes on K , and since $\mathcal{H}_1 \cong (P \otimes \bar{Q})/K$, φ can be lifted to a map starting at \mathcal{H}_1 , $\varphi' : \mathcal{H}_1 \rightarrow \mathcal{H}_{\text{bulk}}$. For $x \in P \otimes \bar{Q}$ and $[x]$ the corresponding class in \mathcal{H}_1 we can write

$$\underline{b}_{\text{disc}} \circ \varphi'([x]) = \underline{b}_{\text{disc}} \circ \varphi(x) = \underline{b}_{\text{disc}} \circ \pi \circ \hat{\varphi}(x) = \underline{b}_{\text{ev}} \circ \hat{\varphi}(x) = \underline{d}(x) = \underline{b}_1([x]). \quad (3.19)$$

This shows that φ' is an intertwiner such that (3.1) commutes. As mentioned below (3.1) this already implies that φ' is unique and injective.

Altogether we have therefore shown that $(\mathcal{H}_{\text{bulk}}, b_{\text{disc}})$ is indeed maximal.

4. The bulk space of the $c_{1,p}$ triplet models

In this section, we want to apply the abstract construction of the previous section to the case of the $c_{1,p}$ triplet models. We begin by collecting some basic properties of the representation theory of the $c_{1,p}$ triplet algebra \mathcal{W}_p .

4.1. Representations of the \mathcal{W}_p -algebra

The representation theory of the \mathcal{W}_p -algebra [37] has been analysed in [2, 3, 29, 31, 32, 38, 59–61]. Let us briefly summarise the aspects we will need in the following.

For a given $p \in \mathbb{Z}_{\geq 2}$ the central charge of the \mathcal{W}_p -algebra is $c = 13 - 6p - 6/p$. The \mathcal{W}_p -algebra has $2p$ irreducible representations that we shall label as

$$U_s^\epsilon, \quad s = 1, \dots, p \quad \epsilon = \pm. \quad (4.1)$$

Here U_1^+ is the vacuum representation and U_1^- describes the simple current. For $p = 2$, U_1^- is the representation \mathcal{V}_1 of [38] and the irreducible representations U_2^\pm are $\mathcal{V}_{-1/8}$ and $\mathcal{V}_{3/8}$.

The projective cover of U_s^ϵ is denoted by P_s^ϵ ; for $s = p$, $U_p^\pm = P_p^\pm$, while for $s = 1, \dots, p - 1$ we have the maximal inclusions

$$U_s^\epsilon \subset M_{+,s}^\epsilon \subset N_s^\epsilon \subset P_s^\epsilon \quad \text{and} \quad U_s^\epsilon \subset M_{-,s}^\epsilon \subset N_s^\epsilon \subset P_s^\epsilon, \quad (4.2)$$

where furthermore $U_s^\epsilon = M_{+,s}^\epsilon \cap M_{-,s}^\epsilon$. Here $M_{\pm,s}^\epsilon$, N_s^ϵ and P_s^ϵ are indecomposable representations⁴. The \mathcal{W}_p -algebra has an infinite number of distinct indecomposable representations [31], but only those mentioned above will be needed in our analysis.

For the case of the $c = -2$ triplet model at $p = 2$, the two indecomposable representations \mathcal{R}_0 and \mathcal{R}_1 of [38] correspond to P_1^\pm . Furthermore, in the symplectic fermion language (we are using the conventions of [40]), the intermediate representations $M_{\pm,1}^\pm$ and N_1^ϵ can be easily described: if we denote the subspace that is generated from $\chi_0^\pm \omega$ by the action of the fermionic modes by $M_{\pm,1}$ then $M_{\pm,1}^+$ denotes the bosonic states in $M_{\pm,1}$, while $M_{\pm,1}^-$ are the fermionic states. Both subspaces form then representations of the (bosonic) triplet algebra. On the other hand, N_1^ϵ is the bosonic ($\epsilon = +$) or fermionic ($\epsilon = -$) subspace generated by $\chi_0^+ \omega$ and $\chi_0^- \omega$ together.

4.2. Intertwiners

For the following it is important to understand the space of intertwiners between two (indecomposable) projective representations P_s^ϵ and P_t^ν . To this end we consider the exact sequences (see appendix C)

$$0 \rightarrow N_s^\epsilon \rightarrow P_s^\epsilon \rightarrow U_s^\epsilon \rightarrow 0, \quad 0 \rightarrow M_{v,s}^\epsilon \rightarrow P_s^\epsilon \rightarrow M_{v,p-s}^{-\epsilon} \rightarrow 0, \quad (4.3)$$

where $s = 1, \dots, p - 1$ and $\epsilon, \nu = \pm$. Together with (4.2) it follows that $\text{Hom}_{\mathcal{V}}(P_s^\epsilon, P_s^\epsilon)$ contains at least two linearly independent maps, namely the identity id, and

$$n : P_s^\epsilon \rightarrow U_s^\epsilon \rightarrow P_s^\epsilon, \quad (4.4)$$

where the intermediate maps are the surjection of the projective cover and the embedding $U_s^\epsilon \subset P_s^\epsilon$. In the symplectic fermion language for $p = 2$, the intertwiner n is simply $n = \chi_0^- \chi_0^+$.

Similarly, the intertwiners $\text{Hom}_{\mathcal{V}}(P_s^\epsilon, P_{p-s}^{-\epsilon})$ contain (we are suppressing the dependence on s and ϵ in the definition of e_ν)

$$e_\nu : P_s^\epsilon \rightarrow M_{v,p-s}^{-\epsilon} \rightarrow P_{p-s}^{-\epsilon} \quad \nu = \pm, \quad (4.5)$$

where the intermediate maps are those appearing in (4.3). Again for $p = 2$, we simply have $e_\pm = \chi_0^\pm$. It is argued in appendix C that the identity map id, (4.4) and (4.5) already give all intertwiners,

$$\text{Hom}_{\mathcal{V}}(P_s^\epsilon, P_t^\nu) = \begin{cases} \mathbb{C} \text{id} \oplus \mathbb{C} n; & t = s, \nu = \epsilon \\ \mathbb{C} e_+ \oplus \mathbb{C} e_-; & t = p - s, \nu = -\epsilon \\ \{0\}; & \text{otherwise.} \end{cases} \quad (4.6)$$

⁴ We follow the conventions used in [32, section 6]. The relation to [29, section 2] is as follows: $\Lambda(s) = U_s^+$, $\Pi(s) = U_s^-$, $\mathcal{R}_0(s) = P_s^+$, $\mathcal{R}_1(s) = P_{p-s}^-$, as well as $\mathcal{N}_0(s) = N_s^+$, $\mathcal{N}_1(s) = N_{p-s}^-$, $\mathcal{N}_0^\pm(s) = M_{\pm,s}^+$ and $\mathcal{N}_1^\pm(s) = M_{\pm,p-s}^-$.

The dimension of the Hom spaces can also be understood from the following diagrams which describe the sub-quotients of P_s^+ and P_{p-s}^- :

$$P_s^+ : \begin{array}{ccc} & U_s^+ & \\ \swarrow & & \searrow \\ U_{p-s}^- & & U_{p-s}^- \\ \searrow & & \swarrow \\ & U_s^+ & \end{array} , \quad P_{p-s}^- : \begin{array}{ccc} & U_{p-s}^- & \\ \swarrow & & \searrow \\ U_s^+ & & U_s^+ \\ \searrow & & \swarrow \\ & U_{p-s}^- & \end{array} . \quad (4.7)$$

The entries in the first diagram correspond to the inclusions (4.2) in the sense that we have the equivalences $P_s^+/N_s^+ \cong U_s^+$ and $N_s^+/U_s^+ \cong U_{p-s}^- \oplus U_{p-s}^-$. The arrows indicate that, e.g. a representative v of a nonzero $[v] \in P_s^+/N_s^+$ (the copy of U_s^+ at the top of the diagram) can get mapped to N_s^+ , but not vice versa.

The exact sequences (4.3) together with the inclusions (4.2) also imply that, for $v = \pm$,

$$n \circ n = 0, \quad e_v \circ n = 0, \quad n \circ e_v = 0, \quad e_v \circ e_v = 0. \quad (4.8)$$

The combination $e_- \circ e_+$ acting on, say, P_s^+ amounts to the composition

$$P_s^+ \twoheadrightarrow M_{+,p-s}^- \hookrightarrow P_{p-s}^- \twoheadrightarrow M_{-,s}^+ \hookrightarrow P_s^+. \quad (4.9)$$

The kernel of the second surjection is $M_{-,p-s}^-$, so that elements of P_s^+ which get mapped to elements of $M_{+,p-s}^-$ that do not also lie in $M_{-,p-s}^-$, have a nonzero image in P_s^+ . In particular, $e_- \circ e_+ \neq 0$. Since $e_- \circ (e_- \circ e_+) = 0$ we see that $e_- \circ e_+$ is proportional to n . Similarly one can check that $e_+ \circ e_-$ is nonzero and proportional to n . We choose to normalize n such that

$$e_- \circ e_+ = n, \quad e_+ \circ e_- = \lambda n \quad \text{for some } \lambda \in \mathbb{C}^\times. \quad (4.10)$$

For the case of $p = 2$, it follows from the above identifications that $\lambda = -1$. In general λ may also depend on s and ε .

4.3. The construction of the bulk space

We now want to apply the general construction of section 3 to the case of the $c_{1,p}$ triplet models. We shall assume that the theory has a boundary condition for which $\mathcal{H}_{\text{bnd}} = \mathcal{V}$. This is motivated by the analysis of [40] where for the $p = 2$ example the boundary conditions were analysed in detail (using the symplectic fermion description). For $p = 2$ we found that the boundary conditions are labelled by the irreducible representations and that the open string spectrum is simply determined by the corresponding fusion rules. This led to the conjecture that the same structure is also present for the other $c_{1,p}$ triplet models. If this is the case, then the brane associated with the irreducible vacuum representation has indeed $\mathcal{H}_{\text{bnd}} = \mathcal{V}$.

We shall thus assume that such a brane exists and determine the corresponding bulk spectrum following the strategy of section 3. As a consistency check we shall later study the boundary states of the resulting bulk theory (see section 5). We shall find that our bulk theory has indeed boundary states associated with the irreducible representations of the triplet algebra, and that their open string spectra are determined by the fusion rules. This therefore forms a stringent consistency check on our procedure.

With this in mind, all we have to do is to calculate the kernel \mathcal{N}_{ev} of the map $\underline{b}_{\text{ev}}$. Recall from section 3.3 the definition of the spaces H_{kl} , as well as the maps c_{kl} and d_{kl} . It follows from (4.6) that the spaces H_{kl} are only nonzero for $(k, l) = ((s, \varepsilon), (s, \varepsilon))$ with $s = 1, \dots, p$ and $(k, l) = ((s, \varepsilon), (p-s, -\varepsilon))$ for $s = 1, \dots, p-1$, where in both cases $\varepsilon = \pm$. It is therefore enough to consider the kernel of $\underline{b}_{\text{ev}}$ separately for the spaces

$$\hat{\mathcal{H}}_s = (P_s^+ \otimes \bar{P}_s^{**}) \oplus (P_{p-s}^- \otimes \bar{P}_{p-s}^{-*}), \quad (4.11)$$

where $s = 1, \dots, p - 1$. (Since P_p^\pm is already irreducible, the kernel in the summands with $s = p$ is trivial.) The spaces H_{kl} are of the form $H^+ := H_{(s,\varepsilon)(s,\varepsilon)}$ and $H^- := H_{(s,\varepsilon)(p-s,-\varepsilon)}$, where

$$\begin{aligned} H^+ &= \text{Hom}_V(P_s^\varepsilon, P_s^\varepsilon) \otimes \text{Hom}_V(P_s^\varepsilon, P_s^\varepsilon) \oplus \text{Hom}_V(P_s^\varepsilon, P_{p-s}^{-\varepsilon}) \otimes \text{Hom}_V(P_{p-s}^{-\varepsilon}, P_s^\varepsilon), \\ H^- &= \text{Hom}_V(P_s^\varepsilon, P_s^\varepsilon) \otimes \text{Hom}_V(P_s^\varepsilon, P_{p-s}^{-\varepsilon}) \oplus \text{Hom}_V(P_s^\varepsilon, P_{p-s}^{-\varepsilon}) \otimes \text{Hom}_V(P_{p-s}^{-\varepsilon}, P_{p-s}^{-\varepsilon}). \end{aligned} \quad (4.12)$$

An element u of H^+ is a linear combination of the form

$$u = a(\text{id} \otimes \text{id}) + b(n \otimes \text{id}) + c(\text{id} \otimes n) + d(n \otimes n) + \sum_{\mu, \nu = \pm} f^{\mu\nu}(e_\mu \otimes e_\nu). \quad (4.13)$$

Applying $c^+ := c_{(s,\varepsilon)(s,\varepsilon)}$ to u yields $c^+(u) = a \text{id} + (b + c + f^{+-} + \lambda f^{-+})n$. The kernel of c^+ is thus given by

$$\begin{aligned} \ker(c^+) &= \text{span}_{\mathbb{C}}\{(n \otimes n), (e_+ \otimes e_+), (e_- \otimes e_-), (n \otimes \text{id} - \text{id} \otimes n), \\ &\quad (\lambda e_+ \otimes e_- - e_- \otimes e_+), (n \otimes \text{id} - e_+ \otimes e_-)\}. \end{aligned} \quad (4.14)$$

Similarly, an element v of H^- is a linear combination of the form

$$v = \sum_{\nu = \pm} (a^\nu(\text{id} \otimes e_\nu) + b^\nu(e_\nu \otimes \text{id}) + c^\nu(n \otimes e_\nu) + d^\nu(e_\nu \otimes n)). \quad (4.15)$$

Applying $c^- := c_{(s,\varepsilon)(p-s,-\varepsilon)}$ to v gives $c^-(v) = \sum_\nu (a^\nu + b^\nu)e_\nu$, so that

$$\ker(c^-) = \text{span}_{\mathbb{C}}\{(n \otimes e_\nu), (e_\nu \otimes n), (\text{id} \otimes e_\nu - e_\nu \otimes \text{id}) | \nu = \pm\}. \quad (4.16)$$

Using (3.13) the kernel \mathcal{N}_{ev} of $\underline{b}_{\text{ev}}$ is now simply $d^\pm(f)$ for the various generators f of $\ker(c^\pm)$.

This specifies the kernel \mathcal{N}_{ev} completely. However, we do not need to consider the image of all of these maps separately. In fact, the kernel is already generated by the images of the last element of $\ker(c^-)$ with $\nu = \pm$. More precisely, we define the space

$$\mathcal{D}_s = (P_s^+ \otimes \bar{P}_{p-s}^*) \oplus (P_{p-s}^- \otimes \bar{P}_s^{+*}). \quad (4.17)$$

Then

$$\mathcal{K}_s^\nu = (\text{id} \otimes \bar{e}_\nu^* - e_\nu \otimes \bar{\text{id}})\mathcal{D}_s, \quad \nu = \pm \quad (4.18)$$

are subspaces of $\hat{\mathcal{H}}_s$. (Recall that e_ν maps P_s^+ to P_{p-s}^- and vice versa, and it is understood that $(\text{id} \otimes \bar{e}_\nu^* - e_\nu \otimes \bar{\text{id}})$ acts on both summands of \mathcal{D}_s .) We now claim that the kernel of $\underline{b}_{\text{ev}}$ restricted to $\hat{\mathcal{H}}_s$ is simply the span of these two spaces,

$$\mathcal{N}_s = \ker(\underline{b}_{\text{ev}}|_{\hat{\mathcal{H}}_s}) = \text{span}_{\mathbb{C}}(\mathcal{K}_s^+, \mathcal{K}_s^-). \quad (4.19)$$

By construction, it is clear that $\mathcal{K}_s^\pm \subset \mathcal{N}_s$; it only remains to prove that they generate already all of \mathcal{N}_s , i.e. that the images of $d^\pm(f)$ for the various generators f of $\ker(c^\pm)$ lie in a linear combination of states from \mathcal{K}_s^\pm . Let us check this explicitly in two examples; the rest can be seen similarly. To obtain the first generator of $\ker(c^+)$ in (4.14) we consider the composition

$$(\text{id} \otimes \bar{e}_+^* - e_+ \otimes \bar{\text{id}}) \circ (n \otimes \bar{e}_-^*) = n \otimes (\bar{e}_+^* \circ \bar{e}_-^*) - (e_+ \circ n) \otimes \bar{e}_-^* = n \otimes (\bar{e}_- \circ \bar{e}_+)^* = n \otimes \bar{n}^*. \quad (4.20)$$

Similarly, the fourth generator in (4.14) is obtained by taking

$$\begin{aligned} &-(\text{id} \otimes \bar{e}_+^* - e_+ \otimes \bar{\text{id}}) \circ (\text{id} \otimes \bar{e}_-^*) - (\text{id} \otimes \bar{e}_-^* - e_- \otimes \bar{\text{id}}) \circ (e_+ \otimes \bar{\text{id}}) \\ &= -\text{id} \otimes (\bar{e}_+^* \circ \bar{e}_-^*) + e_+ \otimes \bar{e}_-^* - e_+ \otimes \bar{e}_-^* + (e_- \circ e_+) \otimes \bar{\text{id}} = -\text{id} \otimes \bar{n}^* + n \otimes \bar{\text{id}}. \end{aligned} \quad (4.21)$$

4.4. The bulk space and comparison to $p = 2$

Summarizing the above discussion we therefore find that the actual bulk space of states $\mathcal{H}_{\text{bulk}}$ is of the form

$$\mathcal{H}_{\text{bulk}} = \bigoplus_{s=1}^{p-1} \hat{\mathcal{H}}_s / \mathcal{N}_s \oplus (U_p^+ \otimes \bar{U}_p^{+*}) \oplus (U_p^- \otimes \bar{U}_p^{-*}), \quad (4.22)$$

where \mathcal{N}_s is defined in (4.19). For $p = 2$ equation (4.22) becomes in the notation of [40]

$$\mathcal{H}_{\text{bulk}} = \mathcal{H}_{\omega}^{\text{bos}} / \mathcal{N} \oplus (\mathcal{V}_{-1/8} \otimes \bar{\mathcal{V}}_{-1/8}) \oplus (\mathcal{V}_{3/8} \otimes \bar{\mathcal{V}}_{3/8}), \quad (4.23)$$

where $\mathcal{H}_{\omega}^{\text{bos}} = (\mathcal{R}_0 \otimes \bar{\mathcal{R}}_0) \oplus (\mathcal{R}_1 \otimes \bar{\mathcal{R}}_1)$. Furthermore, it follows from (4.19) together with the identification of $e_{\pm} = \chi_0^{\pm}$ that \mathcal{N} consists of the states

$$\mathcal{N} = \text{span}_{\mathbb{C}} \{ (\chi_0^{\nu} - \bar{\chi}_0^{\nu}) \psi | \nu = \pm, \psi \in \mathcal{H}_{\omega}^{\text{fer}} \}, \quad (4.24)$$

where $\mathcal{H}_{\omega}^{\text{fer}} = (\mathcal{R}_0 \otimes \bar{\mathcal{R}}_1) \oplus (\mathcal{R}_1 \otimes \bar{\mathcal{R}}_0)$. This then reproduces precisely the description of the bulk theory used in [40].

In [24] harmonic analysis on supergroups was used to obtain the space of bulk states for WZW models with supergroup targets. The similarities in the representation theory of super Lie algebras and the \mathcal{W}_p -algebra were then exploited to propose a description of the bulk space as

$$\mathcal{H}_{\text{bulk}} = \bigoplus_{s=1}^{p-1} \mathcal{I}_s \oplus (U_p^+ \otimes \bar{U}_p^{+*}) \oplus (U_p^- \otimes \bar{U}_p^{-*}), \quad (4.25)$$

and a composition series for \mathcal{I}_s was given. (However, unlike the expression (4.22) in terms of quotients, a composition series does in general not fix a representation up to isomorphism.) In [24] it is also conjectured that as a $\bar{\mathcal{W}}_p$ -representation (but not as a $\mathcal{W}_p \times \bar{\mathcal{W}}_p$ -representation) \mathcal{I}_s is of the form

$$\mathcal{I}_s = (U_s^+ \otimes \bar{P}_s^{+*}) \oplus (U_{p-s}^- \otimes \bar{P}_{p-s}^{-*}). \quad (4.26)$$

In order to compare this prediction with our result we now have to decompose our quotient space $\mathcal{H}_s \equiv \hat{\mathcal{H}}_s / \mathcal{N}_s$ with respect to the $\bar{\mathcal{W}}_p$ action. This is done in appendix D and we find agreement with (4.26). It is encouraging that the two proposals fit together.

4.5. Modular invariance for the $c_{1,p}$ triplet models

Finally, we want to show that the partition function of the bulk space $\mathcal{H}_{\text{bulk}}$ is modular invariant. We have already proven in section 3.2 that the partition function is invariant under $\tau \mapsto \tau + 1$. This followed from the fact that $\exp(2\pi i(L_0 - \bar{L}_0))$ acts as the identity on $\mathcal{H}_{\text{bulk}}$. In the present context this can be seen more concretely because the element $n \otimes \text{id} - \text{id} \otimes n$ is in $\ker(c^+)$, see equation (4.14), and since $e^{2\pi i L_0} \in \text{Hom}_{\mathbb{V}}(P_s^{\varepsilon}, P_s^{\varepsilon})$ can be written as a linear combination of id and n .

With the help of equation (4.26) it is now straightforward to compute the partition function of $\mathcal{H}_{\text{bulk}}$ in (4.22). We will start by recalling the expressions for the characters of the \mathcal{W}_p -representations and their modular properties. These were first described in [2]; here we will follow the presentation in [29, section 3]. The characters of the irreducible representations are, for $s = 1, \dots, p$ and $\nu = \pm$,

$$\chi_{U_s^+}(\tau) = \frac{1}{\eta(q)} \left(\frac{s}{p} \theta_{p-s,p}(q) + 2\theta'_{p-s,p}(q) \right), \quad \chi_{U_s^-}(\tau) = \frac{1}{\eta(q)} \left(\frac{s}{p} \theta_{s,p}(q) - 2\theta'_{s,p}(q) \right). \quad (4.27)$$

Here $\eta(q)$ is the Dedekind eta function and $\theta_{s,p}(q) = \theta_{s,p}(1, q)$ with

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad \theta_{s,p}(z, q) = \sum_{m \in \mathbb{Z} + \frac{s}{2p}} z^m q^{pm^2}. \quad (4.28)$$

We also define

$$\theta'_{s,p}(q) = z \frac{\partial}{\partial z} \theta_{s,p}(z, q) \Big|_{z=1}. \quad (4.29)$$

Both $\theta_{s,p}$ and $\theta'_{s,p}$ are periodic in s with period $2p$. In addition we have $\theta_{s,p}(q) = \theta_{2p-s,p}(q)$ and $\theta'_{s,p}(q) = -\theta'_{2p-s,p}(q)$. Thus we can restrict s to take the values $s = 0, \dots, p$. Furthermore it follows that $\theta'_{0,p}(q) = 0 = \theta'_{p,p}(q)$. The modular transformation properties under $\tau \mapsto -1/\tau$ are

$$\begin{aligned} \frac{\theta_{s,p}}{\eta} \left(-\frac{1}{\tau} \right) &= \frac{1}{\sqrt{2p}} \sum_{s'=0}^{2p-1} e^{i\pi s s' / p} \frac{\theta_{s',p}}{\eta}(\tau) \\ &= \frac{1}{\sqrt{2p}} \left[\frac{\theta_{0,p}}{\eta}(\tau) + (-1)^s \frac{\theta_{p,p}}{\eta}(\tau) + 2 \sum_{s'=1}^{p-1} \cos\left(\frac{\pi s s'}{p}\right) \frac{\theta_{s',p}}{\eta}(\tau) \right], \end{aligned} \quad (4.30)$$

and similarly

$$\frac{\theta'_{s,p}}{\eta} \left(-\frac{1}{\tau} \right) = -\frac{\tau}{\sqrt{2p}} \sum_{s'=0}^{2p-1} e^{i\pi s s' / p} \frac{\theta'_{s',p}}{\eta}(\tau) = -\frac{2i\tau}{\sqrt{2p}} \sum_{s'=1}^{p-1} \sin\left(\frac{\pi s s'}{p}\right) \frac{\theta'_{s',p}}{\eta}(\tau). \quad (4.31)$$

For the following it is also useful to abbreviate, for $s = 1, \dots, p - 1$,

$$\psi_s^+(\tau) = \chi_{U_s^+}(\tau) + \chi_{U_{p-s}^-}(\tau) = \frac{\theta_{p-s,p}(q)}{\eta(q)}, \quad \psi_s^-(\tau) = \chi_{U_s^-}(\tau) + \chi_{U_{p-s}^+}(\tau) = \frac{\theta_{s,p}(q)}{\eta(q)}. \quad (4.32)$$

It then follows from the exact sequences in section 4.2 that $\text{tr}_{P_s^\pm}(q^{L_0-c/24}) = 2\psi_s^\pm(\tau)$ for $s = 1, \dots, p - 1$. Equation (4.26) now implies that

$$\text{tr}_{\mathcal{H}_s}(q^{L_0-c/24}(q^*)^{\bar{L}_0-c/24}) = 2\chi_{U_s^+}(\tau) \overline{\psi_s^+(\tau)} + 2\chi_{U_{p-s}^-}(\tau) \overline{\psi_{p-s}^-(\tau)} = 2|\psi_s^+(\tau)|^2, \quad (4.33)$$

where we have used that $\psi_{p-s}^- = \psi_s^+$. Taking the trace of $\mathcal{H}_{\text{bulk}}$ and using the previously mentioned relations between the different $\theta_{s,p}$ functions then gives

$$\begin{aligned} Z(\tau) &= \text{tr}_{\mathcal{H}_{\text{bulk}}}(q^{L_0-c/24}(q^*)^{\bar{L}_0-c/24}) \\ &= |\chi_{U_p^+}(\tau)|^2 + |\chi_{U_p^-}(\tau)|^2 + 2 \sum_{s=1}^{p-1} |\psi_s^+(\tau)|^2 = \frac{1}{|\eta(\tau)|^2} \sum_{s=0}^{2p-1} |\theta_{s,p}(q)|^2. \end{aligned} \quad (4.34)$$

As already noted in [2] (see also [24]), the last expression for $Z(\tau)$ is easily checked to obey $Z(-1/\tau) = Z(\tau)$ using the modular properties of the theta functions.

5. Boundary states

Now that we have the bulk spectrum under control we can analyse the possible Ishibashi states and construct the boundary states. This will be a consistency check of our approach since we started out by assuming that the theory possesses an ‘identity brane’ whose open string spectrum only consists of the chiral algebra itself.

5.1. The space of Ishibashi states

Boundary conditions are usually described in terms of boundary states that live in a suitable completion of $\mathcal{H}_{\text{bulk}}$. Instead of giving the boundary state explicitly, we may also specify it by giving all bulk one-point functions on the disc (or on the upper half plane). Thus we may think of a boundary condition as being described by a linear form $\mathcal{H}_{\text{bulk}} \rightarrow \mathbb{C}$. For concreteness, we take this linear form to be defined by inserting the bulk field on the upper half plane at the point $z = i$.

Every boundary condition that preserves the symmetry described by \mathcal{V} necessarily contains \mathcal{V} as a subspace of \mathcal{H}_{bnd} , and we may as well consider the correlator of a bulk field inserted at $z = i$, together with a boundary field in the subspace \mathcal{V} inserted at 0. Thus we are led to consider a bilinear map $\mathcal{H}_{\text{bulk}} \times \mathcal{V} \rightarrow \mathbb{C}$. The compatibility conditions with the action of \mathcal{V} for such a bilinear map are just the same as for b_{disc} . After all, b_{disc} is precisely such a correlator for the identity brane. The space of bilinear maps $\mathcal{H}_{\text{bulk}} \times \mathcal{V} \rightarrow \mathbb{C}$ compatible with the action of \mathcal{V} is by definition the space of Ishibashi states, and we therefore obtain⁵

$$B(\mathcal{H}_{\text{bulk}}) \text{ is the space of Ishibashi states for } \mathcal{H}_{\text{bulk}}. \tag{5.1}$$

Here $\mathcal{H}_{\text{bulk}}$ is the quotient (3.4) and $B(\mathcal{H}_{\text{bulk}})$ is defined as in section 3.1 and appendix A.

Since $\mathcal{H}_{\text{bulk}}$ is defined as the quotient $\hat{\mathcal{H}}/\mathcal{N}_{\text{ev}}$, $B(\mathcal{H}_{\text{bulk}})$ is isomorphic to the space of linear forms on $\hat{\mathcal{H}}$ that lie in $B(\hat{\mathcal{H}})$ and that vanish on \mathcal{N}_{ev} ,

$$B(\mathcal{H}_{\text{bulk}}) \cong \{b \in B(\hat{\mathcal{H}}) | b(v) = 0 \text{ for all } v \in \mathcal{N}_{\text{ev}}\}. \tag{5.2}$$

Because of (3.5) and (3.6) every element $b \in B(\hat{\mathcal{H}})$ can be written as

$$b(p \otimes \bar{q}, v) = \sum_{k \in \text{Irr}} \text{ev}_{P_k}(\rho_k(p), q, v) \tag{5.3}$$

for an appropriate $\rho_k \in \text{Hom}_{\mathcal{V}}(P_k, P_k)$. In fact, this defines an isomorphism

$$\bigoplus_{k \in \text{Irr}} \text{Hom}_{\mathcal{V}}(P_k, P_k) \xrightarrow{\cong} B(\hat{\mathcal{H}}). \tag{5.4}$$

By the same arguments used to obtain the description of \mathcal{N}_{ev} in (3.13), it is not hard to see that \underline{b} vanishes on \mathcal{N}_{ev} if and only if $\sum_i g_i \circ \rho_i \circ f_i = 0$ whenever $\sum_i g_i \circ f_i = 0$, i.e. whenever $\sum_i g_i \otimes f_i$ lies in the kernel of c_{kl} as defined in (3.9). But let us see more concretely what the result is for the $c_{1,p}$ triplet models.

For the case of the $c_{1,p}$ triplet models, we can consider the various summands of $\mathcal{H}_{\text{bulk}}$ in (4.22) separately. The two irreducible summands corresponding to P_p^\pm give rise to two Ishibashi states corresponding to $\rho = \text{id}_{P_p^+}$ and $\rho = \text{id}_{P_p^-}$, respectively. (These are the familiar Ishibashi states associated with irreducible representations.) The situation is more interesting for $\mathcal{H}_s = \hat{\mathcal{H}}_s/\mathcal{N}_s$ with $s = 1, \dots, p-1$. The space of intertwiners

$$(\rho_+, \rho_-) \in \text{Hom}_{\mathcal{V}}(P_s^+, P_s^+) \oplus \text{Hom}_{\mathcal{V}}(P_{p-s}^-, P_{p-s}^-) \tag{5.5}$$

is four dimensional, but we also have to impose the condition that (ρ_+, ρ_-) vanishes on \mathcal{N}_s . Since \mathcal{N}_s is generated by \mathcal{K}_s^\pm , we thus need to analyse whether (ρ_+, ρ_-) vanishes on \mathcal{K}_s^\pm . Using (3.6) this leads to the condition

$$\begin{aligned} e_v \circ \rho_+ - \rho_- \circ e_v &= 0 \in \text{Hom}_{\mathcal{V}}(P_s^+, P_{p-s}^-) & \text{and} \\ e_v \circ \rho_- - \rho_+ \circ e_v &= 0 \in \text{Hom}_{\mathcal{V}}(P_{p-s}^-, P_s^+), \end{aligned} \tag{5.6}$$

⁵ The term Ishibashi state is more commonly used for a state in $\mathcal{H}_{\text{bulk}}$ and appears in the description of boundaries without insertions of \mathcal{V} . The two descriptions are equivalent. For example, given a $b \in B(\mathcal{H}_{\text{bulk}})$, the corresponding element $\phi \in \mathcal{H}_{\text{bulk}}$ is determined by $b(\phi', \Omega) = \langle \phi(-i)\phi'(i) \rangle$ for all $\phi' \in \mathcal{H}_{\text{bulk}}$. Here the right-hand side is the bulk two-point function on the complex plane and $\Omega \in \mathcal{V}$ is the vacuum vector of the vertex algebra. (To obtain boundary states for the disc, rather than the upper half plane, one should employ an appropriate conformal transformation.)

where $\nu = \pm$. A short calculation using (4.8) shows that the space of solutions is three dimensional and is given by

$$\rho_+ = \alpha \text{id} + \beta n, \quad \rho_- = \alpha \text{id} + \gamma n, \quad (5.7)$$

where $\alpha, \beta, \gamma \in \mathbb{C}$. (In the first equation id and n act on P_s^+ , while in the second they act on P_{p-s}^- .) Altogether we therefore obtain $3(p-1)$ Ishibashi states from the indecomposable sectors \mathcal{H}_s , as well as two Ishibashi states from the irreducible representations, giving in total $3p-1$ Ishibashi states. Obviously this agrees with the explicit analysis for $p=2$ in [40]. It also agrees with the number of chiral torus amplitudes [62], as may have been expected.

As in the case for $p=2$ we do not expect that all of these Ishibashi states will contribute to the boundary states. Indeed, the space of torus amplitudes contains only $2p$ functions that are power series in q , while the remaining $p-1$ torus functions involve terms proportional to τ [62] (see also [2]). The latter cannot appear in a consistent open string expansion (since the open string description involves a trace that can never lead to a term proportional to τ), and thus the space of open string amplitudes is only $2p$ dimensional. But then it follows that only a $2p$ -dimensional subspace of the Ishibashi states can contribute to consistent boundary states. In fact, one would expect that for each \mathcal{H}_s only two linear combinations of the three Ishibashi states can contribute. This expectation is borne out by the detailed construction to which we now turn.

5.2. Constructing the boundary states

The analysis of [40] suggests that the boundary states of the ‘charge-conjugation’ $c_{1,p}$ triplet models are labelled by the irreducible representations of the \mathcal{W}_p -algebra. As we have explained before, the irreducible representations are labelled by (s, ε) , where $s = 1, \dots, p$ and $\varepsilon = \pm$; we shall denote the corresponding boundary states as $\|(s, \varepsilon)\rangle\rangle$. Given the results of [40] it is furthermore natural to expect that their open string spectrum is described by the fusion rules, i.e. that

$$\langle\langle (s_1, \varepsilon_1) \| q^{\frac{1}{2}(L_0 + \bar{L}_0) - \frac{c}{24}} \| (s_2, \varepsilon_2) \rangle\rangle = \sum_{\mathcal{R}} \mathcal{N}_{(s_1, \varepsilon_1)(s_2, \varepsilon_2)}^{\text{fus}} \mathcal{R} \text{tr}_{\mathcal{R}}(\tilde{q}^{L_0 - c/24}), \quad (5.8)$$

where as always in the following $q = e^{2\pi i \tau}$ and $\tilde{q} = e^{-2\pi i/\tau}$. Here $\mathcal{N}_{(s_1, \varepsilon_1)(s_2, \varepsilon_2)}^{\text{fus}} \mathcal{R}$ gives the decomposition of the fusion product of two irreducible representations,

$$U_{s_1}^{\varepsilon_1} \otimes U_{s_2}^{\varepsilon_2} = \bigoplus_{\mathcal{R}} \mathcal{N}_{(s_1, \varepsilon_1)(s_2, \varepsilon_2)}^{\text{fus}} \mathcal{R}, \quad (5.9)$$

for more details see appendix C. The ansatz (5.8) is the natural generalization of the usual Cardy situation [56] to rational logarithmic conformal field theories.

On the level of characters one cannot tell the difference between P_s^v and $2U_s^v \oplus 2U_{p-s}^{-v}$, so that we may as well write (5.8) directly in terms of the structure constants N of the Grothendieck ring, also given in appendix C,

$$\langle\langle (s_1, \varepsilon_1) \| q^{\frac{1}{2}(L_0 + \bar{L}_0) - \frac{c}{24}} \| (s_2, \varepsilon_2) \rangle\rangle = \sum_{r=1}^p \sum_{\mu=\pm} N_{(s_1, \varepsilon_1)(s_2, \varepsilon_2)}^{(r, \mu)} \chi_{U_r^\mu}(\tilde{q}). \quad (5.10)$$

Starting from the ansatz (5.10) we now want to construct the boundary states explicitly. To this end, we will evaluate (5.10) in the special case $(s_1, \varepsilon_1) = (1, +)$ and (s_2, ε_2) arbitrary, as well as for $(s_1, \varepsilon_1) = (s_2, \varepsilon_2) = (p, +)$. This will determine the boundary states $\|(s, \varepsilon)\rangle\rangle$ and the overlap of the Ishibashi states. It is then a highly non-trivial consistency check that all other overlaps also agree with (5.10).

To carry out this calculation it is convenient to work with the S -matrices of the \mathcal{W}_p -characters. These can be found from the transformation properties of the theta-functions in section 4.5 or by rearranging [29, prop. 3.4],

$$\chi_{U_s^\varepsilon}(\tilde{q}) = \sum_{t=1}^p \sum_{v=\pm} (S_{(s,\varepsilon),(t,v)} - i\tau S_{(s,\varepsilon),(t,v)}^\ell) \chi_{U_t^v}(q). \quad (5.11)$$

Here $\tilde{q} = e^{-2\pi i/\tau}$ is the open string loop parameter, while $q = e^{2\pi i\tau}$ is the corresponding parameter in the closed string. The matrices S and S^ℓ are, for $s, t = 1, \dots, p$ and $\varepsilon, v = \pm$,

$$\begin{aligned} S_{(s,\varepsilon),(t,v)} &= \frac{2 - \delta_{t,p}}{\sqrt{2p}} \frac{s}{p} \cos\left(\pi \frac{st}{p}\right) (-v)^s (-\varepsilon)^t (-1)^{p(\varepsilon+1)(v+1)/4}, \\ S_{(s,\varepsilon),(t,v)}^\ell &= \frac{2}{\sqrt{2p}} \frac{p-t}{p} \sin\left(\pi \frac{st}{p}\right) (-v)^s (-\varepsilon)^t (-1)^{p(\varepsilon+1)(v+1)/4}. \end{aligned} \quad (5.12)$$

Note that $S_{(s,\varepsilon),(p,v)}^\ell = 0$ and that for $t = 1, \dots, p-1$ these S -matrices have the symmetries

$$S_{(s,\varepsilon),(p-t,-v)} = S_{(s,\varepsilon),(t,v)}, \quad (p-t)S_{(s,\varepsilon),(p-t,-v)}^\ell = -tS_{(s,\varepsilon),(t,v)}^\ell. \quad (5.13)$$

With the help of these identities we can rewrite (5.11) in the following form:

$$\chi_{U_s^\varepsilon}(\tilde{q}) = \sum_{v=\pm} S_{(s,\varepsilon),(p,v)} \chi_{U_p^v}(q) + \sum_{t=1}^{p-1} (S_{(s,\varepsilon),(t,+)} \psi_t^+(q) + S_{(s,\varepsilon),(t,+)}^\ell \varphi_t^+(q)), \quad (5.14)$$

where $\psi_s^+(q)$ was defined in (4.32) and $\varphi_s^+(q)$ is given by

$$\varphi_s^+(q) = -i\tau \frac{2p}{p-s} \left(\frac{p-s}{2p} \chi_{U_s^+}(q) - \frac{s}{2p} \chi_{U_{p-s}^-}(q) \right) = -i\tau \frac{2p}{p-s} \frac{\theta'_{p-s,p}(q)}{\eta(q)}. \quad (5.15)$$

To determine the boundary states we begin by considering the overlap of the brane associated with the vacuum representation $(1, +)$ with itself. Since the fusion of the vacuum with the vacuum is just the vacuum we have

$$\begin{aligned} \langle\langle (1, +) | q^{\frac{1}{2}(L_0 + \bar{L}_0) - \frac{c}{24}} | (1, +) \rangle\rangle &= \chi_{U_1^+}(\tilde{q}) \\ &= \sum_{v=\pm} S_{(1,+),(p,v)} \chi_{U_p^v}(q) + \sum_{t=1}^{p-1} (S_{(1,+),(t,+)} \psi_t^+(q) + S_{(1,+),(t,+)}^\ell \varphi_t^+(q)). \end{aligned} \quad (5.16)$$

The terms from the first sum of the last line come from the Ishibashi states in the sectors U_p^\pm , while the contributions with $t = 1, \dots, p-1$ come from Ishibashi states in \mathcal{H}_t . The former Ishibashi states $|U_p^\pm\rangle\rangle$ are unique up to normalization, and we choose

$$\langle\langle U_p^v | q^{\frac{1}{2}(L_0 + \bar{L}_0) - \frac{c}{24}} | U_p^v \rangle\rangle = S_{(1,+),(p,v)} \chi_{U_p^v}(q), \quad (5.17)$$

where the additional factor avoids the introduction of square roots later on. As regards the Ishibashi states coming from \mathcal{H}_t , the analysis in section 5.1 showed that there are *a priori* three independent such Ishibashi states, but it will turn out that we will need only two. Given the validity of (5.16) there has to exist an Ishibashi state $|P_t\rangle\rangle$ in \mathcal{H}_t such that

$$\langle\langle P_t | q^{\frac{1}{2}(L_0 + \bar{L}_0) - \frac{c}{24}} | P_t \rangle\rangle = S_{(1,+),(t,+)} \psi_t^+(q) + S_{(1,+),(t,+)}^\ell \varphi_t^+(q). \quad (5.18)$$

Then the boundary state corresponding to the vacuum brane $(1, +)$ is simply

$$\langle\langle (1, +) \rangle\rangle = |U_p^+\rangle\rangle + |U_p^-\rangle\rangle + \sum_{t=1}^{p-1} |P_t\rangle\rangle. \quad (5.19)$$

For the general boundary state $\|(s, \varepsilon)\rangle\rangle$ we now make the ansatz

$$\|(s, \varepsilon)\rangle\rangle = B_{(s,\varepsilon)}^+ |U_p^+\rangle\rangle + B_{(s,\varepsilon)}^- |U_p^-\rangle\rangle + \sum_{t=1}^{p-1} (B_{(s,\varepsilon)}^t |P_t\rangle\rangle + \tilde{B}_{(s,\varepsilon)}^t |U_t\rangle\rangle), \quad (5.20)$$

where $|U_t\rangle\rangle$ is a second Ishibashi state in \mathcal{H}_t whose overlap with $|P_t\rangle\rangle$ will be determined shortly. By our ansatz (5.10) we must have

$$\begin{aligned} \langle\langle (1, +) | q^{\frac{1}{2}(L_0 + \bar{L}_0) - \frac{c}{24}} \|(s, \varepsilon)\rangle\rangle &= \chi_{U_s^\varepsilon}(\tilde{q}) \\ &= \sum_{v=\pm} S_{(s,\varepsilon),(p,v)} \chi_{U_p^v}(q) + \sum_{t=1}^{p-1} (S_{(s,\varepsilon),(t,+)} \psi_t^+(q) + S_{(s,\varepsilon),(t,+)}^\ell \varphi_t^+(q)). \end{aligned} \quad (5.21)$$

This shows that the overlap of $|U_t\rangle\rangle$ and $|P_t\rangle\rangle$ has to be a linear combination of $\psi_t^+(q)$ and $\varphi_t^+(q)$. For $t = 1, \dots, p-1$ we have $S_{(1,+),(t,+)}^\ell \neq 0$ so that the second term in (5.18) is always nonzero⁶. Thus by redefining $|U_t\rangle\rangle \mapsto |U_t\rangle\rangle + \lambda |P_t\rangle\rangle$ if necessary, and by rescaling $|U_t\rangle\rangle$, we can achieve that

$$\langle\langle P_t | q^{\frac{1}{2}(L_0 + \bar{L}_0) - \frac{c}{24}} |U_t\rangle\rangle = \psi_t^+(q). \quad (5.22)$$

Comparing (5.20) and (5.21) now results in the conditions $S_{(1,+),(p,v)} B_{(s,\varepsilon)}^v = S_{(s,\varepsilon),(p,v)}$ as well as

$$S_{(1,+),(t,+)}^\ell B_{(s,\varepsilon)}^t = S_{(s,\varepsilon),(t,+)}^\ell, \quad S_{(1,+),(t,+)} B_{(s,\varepsilon)}^t + \tilde{B}_{(s,\varepsilon)}^t = S_{(s,\varepsilon),(t,+)} \quad (5.23)$$

The general boundary state is therefore given by

$$\begin{aligned} \|(s, \varepsilon)\rangle\rangle &= \sum_{v=\pm} \frac{S_{(s,\varepsilon),(p,v)}}{S_{(1,+),(p,v)}} |U_p^v\rangle\rangle + \sum_{t=1}^{p-1} \frac{S_{(s,\varepsilon),(t,+)}^\ell}{S_{(1,+),(t,+)}^\ell} |P_t\rangle\rangle \\ &\quad + \sum_{t=1}^{p-1} \left(S_{(s,\varepsilon),(t,+)} - S_{(1,+),(t,+)} \frac{S_{(s,\varepsilon),(t,+)}^\ell}{S_{(1,+),(t,+)}^\ell} \right) |U_t\rangle\rangle. \end{aligned} \quad (5.24)$$

Before we can verify that this indeed reproduces (5.10) we have to compute the self-overlap of $|U_t\rangle\rangle$. Let us set $\langle\langle U_t | q^{\frac{1}{2}(L_0 + \bar{L}_0) - \frac{c}{24}} |U_t\rangle\rangle = h_t(q)$. Then from (5.24) and (5.12) it follows that

$$\begin{aligned} \langle\langle (p, +) | q^{\frac{1}{2}(L_0 + \bar{L}_0) - \frac{c}{24}} \|(p, +)\rangle\rangle &= \sqrt{2/p} (\chi_{U_p^+}(q) - (-1)^p \chi_{U_p^-}(q)) + \sum_{t=1}^{p-1} \frac{2}{p} h_t(q) \\ &= \sum_{t=1}^{p-1} \frac{2}{p} h_t(q) + \begin{cases} 2 \sum_{r=1;2}^{p-1} (\chi_{U_r^+}(\tilde{q}) + \chi_{U_{p-r}^-}(\tilde{q})) & \text{if } p \text{ even,} \\ \chi_{U_p^+}(\tilde{q}) + 2 \sum_{r=1;2}^{p-2} (\chi_{U_r^+}(\tilde{q}) + \chi_{U_{p-r}^-}(\tilde{q})) & \text{if } p \text{ odd,} \end{cases} \end{aligned} \quad (5.25)$$

where the ‘;2’ in the sum means that the sum is taken in steps of two. This should be equal to (5.10) for $(s_1, \varepsilon_1) = (s_2, \varepsilon_2) = (p, +)$. Note that the fusion of U_p^+ with itself is

$$U_p^+ \otimes U_p^+ = \begin{cases} P_1^+ \oplus P_3^+ \oplus \dots \oplus P_{p-1}^+ & \text{if } p \text{ even} \\ P_1^+ \oplus P_3^+ \oplus \dots \oplus P_{p-2}^+ \oplus U_p^+ & \text{if } p \text{ odd} \end{cases} \quad (5.26)$$

(see appendix C). Comparing the character of this and (5.25) fixes $h_t(q) = 0$, so that

$$\langle\langle U_t | q^{\frac{1}{2}(L_0 + \bar{L}_0) - \frac{c}{24}} |U_t\rangle\rangle = 0. \quad (5.27)$$

⁶ Note that, on the other hand, $S_{(1,+),(p/2,+)} = 0$ for p even.

Now that the overlaps of the Ishibashi states and the boundary states are fixed we can perform the consistency check of our ansatz by substituting (5.24) into (5.10). This results in the expression

$$\begin{aligned}
 N_{(s_1, \varepsilon_1)(s_2, \varepsilon_2)}^{(r, \mu)} &= \sum_{v=\pm} \frac{S_{(s_1, \varepsilon_1), (p, v)} S_{(s_2, \varepsilon_2), (p, v)} S_{(p, v), (r, \mu)}}{S_{(1, +), (p, v)}} \\
 &+ \sum_{t=1}^{p-1} \sum_{v=\pm} \left(\frac{S_{(s_1, \varepsilon_1), (t, v)} S_{(s_2, \varepsilon_2), (t, v)}^\ell + S_{(s_1, \varepsilon_1), (t, v)}^\ell S_{(s_2, \varepsilon_2), (t, v)}}{S_{(1, +), (t, v)}^\ell} \right. \\
 &\quad \left. - \frac{S_{(s_1, \varepsilon_1), (t, v)}^\ell S_{(s_2, \varepsilon_2), (t, v)}^\ell (\mathbf{i}\tau + S_{(1, +), (t, v)} / S_{(1, +), (t, v)}^\ell)}{S_{(1, +), (t, v)}^\ell} \right) \\
 &\times \left(S_{(t, v), (r, \mu)} - \mathbf{i} \left(\frac{-1}{\tau} \right) S_{(t, v), (r, \mu)}^\ell \right). \tag{5.28}
 \end{aligned}$$

In particular, the τ -dependence on the right-hand side has to cancel. We have verified numerically in a large number of examples that the right-hand side of (5.28) indeed reproduces the structure constants of the Grothendieck ring as determined by (C.3).

While equation (5.28) still looks quite complicated, it can be simplified considerably in the following way. For $t = 1, \dots, p - 1$ consider the $S_{(s, \varepsilon), (t, v)}$ and $S_{(s, \varepsilon), (t, v)}^\ell$ as formal variables and introduce a derivation D on these by setting $D[S^\ell] = S$ and $D[S] = 0$, i.e.

$$D[f(S_{(s, \varepsilon), (t, v)}^\ell)] = f'(S_{(s, \varepsilon), (t, v)}^\ell) S_{(s, \varepsilon), (t, v)}, \quad D[f(S_{(s, \varepsilon), (t, v)})] = 0. \tag{5.29}$$

Then one can, e.g., write (5.11) as

$$\chi_{U_\varepsilon^r}(\tilde{q}) = \sum_{v=\pm} S_{(s, \varepsilon), (p, v)} \chi_{U_p^v}(q) + \sum_{t=1}^{p-1} \sum_{v=\pm} (D - \mathbf{i}\tau \text{id}) [S_{(s, \varepsilon), (t, v)}^\ell] \cdot \chi_{U_t^v}(q). \tag{5.30}$$

Equation (5.28) can then be written more compactly as

$$\begin{aligned}
 N_{(s_1, \varepsilon_1)(s_2, \varepsilon_2)}^{(r, \mu)} &= \sum_{v=\pm} \frac{S_{(s_1, \varepsilon_1), (p, v)} S_{(s_2, \varepsilon_2), (p, v)} S_{(p, v), (r, \mu)}}{S_{(1, +), (p, v)}} \\
 &+ \sum_{t=1}^{p-1} \sum_{v=\pm} (D - \mathbf{i}\tau \text{id}) \left[\frac{S_{(s_1, \varepsilon_1), (t, v)}^\ell S_{(s_2, \varepsilon_2), (t, v)}^\ell}{S_{(1, +), (t, v)}^\ell} \right] \cdot \left(D - \mathbf{i} \left(\frac{-1}{\tau} \right) \text{id} \right) [S_{(t, v), (r, \mu)}^\ell], \tag{5.31}
 \end{aligned}$$

where the τ -dependent terms on the right-hand side cancel. This formula can be understood as a logarithmic version of the Verlinde formula [63], which describes the structure constants of the Grothendieck ring in non-logarithmic rational conformal field theories (of course, for these the Grothendieck ring coincides with the fusion rules of the irreducible representations). Verlinde-like formulae for \mathcal{W}_p -representations have also been studied in [2, 29, 65].

As opposed to the procedure in [29] resting on block diagonalizing the fusion rules⁷, the formula (5.31) does not involve any choices; the matrices S and S^ℓ are uniquely fixed by (5.11). Furthermore, in the given form it is quite suggestive how to generalize (5.31): the sum runs over all irreducible representations and each summand in (5.31) looks as in the usual Verlinde formula, but with additional D -operators inserted, where the number of insertions is related to the size of the Jordan cell of L_0 in the corresponding projective cover. (For the \mathcal{W}_p -representations these Jordan cells are all of length one or two.)

⁷ The matrix S appearing in the block diagonalization is related to S and S^ℓ via $S = S + S^\ell$ (see [29], equations (3.3), (4.12) and (5.17)).

6. Conclusion and outlook

In this paper, we have proposed a simple method to compute the space of bulk fields for a logarithmic rational conformal field theory. The construction starts from the assumption that there is a boundary condition whose space of boundary fields consists only of the chiral algebra \mathcal{V} itself. The space of bulk fields is then the largest space that can be coupled to the space of boundary fields in a non-degenerate way, consistent with the action of \mathcal{V} . When applied to non-logarithmic rational conformal field theories, this construction yields the charge-conjugation modular invariant theory.

We verified that our method gives a modular invariant partition function when applied to the $c_{1,p}$ triplet models. As a consistency check of the ansatz we computed the set of boundary states—one for each irreducible representation of the \mathcal{W}_p -algebra—and checked that their overlaps give consistent amplitudes in the open channel. We also confirmed that there is indeed a boundary condition whose space of boundary fields is given by \mathcal{V} . The analysis of the boundary states finally led to a Verlinde-like formula for the structure constants of the Grothendieck ring of the \mathcal{W}_p -representations. We also conjectured a formula for the fusion rules of these representations.

There are a number of questions that deserve further study:

- (i) While we were able to show that the partition function is invariant under the T -modular transformation, $Z(\tau) = Z(\tau + 1)$, it remains to prove in general that it is also invariant under the S -transformation, $Z(\tau) = Z(-1/\tau)$.
- (ii) The ansatz for the spaces of boundary fields given in (5.8) does determine the character of the corresponding \mathcal{W}_p -representation, but not the representation itself. On the other hand, the tensor product (C.4) of irreducible representations provides a natural conjecture for the \mathcal{W}_p action on the spaces of boundary fields. For $p = 2$ this has been verified to some extent in [40], and it would be good to check that this remains true for all p .
- (iii) To have a consistent conformal field theory one also has to find a set of structure constants that satisfy the sewing constraints. For non-logarithmic rational theories such a set of structure constants is uniquely determined by the boundary theory [48, 50], and it would be interesting to understand to which extent this remains true for logarithmic models.
- (iv) As already noted in [24, section 6.2], the result (4.22) for the space of bulk states bears a remarkable resemblance to the decomposition of the regular representation of $\bar{U}_q \mathfrak{sl}(2)$ (see [64, prop. 4.4.2]). Understanding this relation better might help to formulate the construction presented in this paper on a purely categorical level without explicit mention of the action of \mathcal{V} .
- (v) The analysis of WZW models with supergroup targets in [21–24] uses quite a different starting point to obtain the bulk space as compared to our construction, namely harmonic analysis on supergroups. It would be interesting to evaluate our quotient expression for $\mathcal{H}_{\text{bulk}}$ (3.4) for these supergroup models, and see if the result agrees with their findings.
- (vi) It would be good to understand the precise relation between [2, 29, 65] and our formula (5.31), in particular since we recover the structure constants of the Grothendieck ring determined in [29]. Also, it would be very interesting to see if (5.31), with the modifications suggested there, determines the Grothendieck ring of other logarithmic conformal field theories as well.

We hope to return to some of these points in the near future.

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Appendix A. Compatibility of b with the \mathcal{V} -action

In this appendix, we describe the invariance condition that defines the space $B(\hat{\mathcal{H}})$ for a given $\mathcal{V} \times \bar{\mathcal{V}}$ -representation $\hat{\mathcal{H}}$. This condition comes directly from the definition of conformal blocks as co-invariants with respect to an action of the conformal vertex algebra (see [66]).

Let W be a quasi-primary element of \mathcal{V} of conformal weight h_W . Let $\mathcal{F}(h_W)$ be the space of functions that are meromorphic on \mathbb{C} , holomorphic on $\mathbb{C} - \{i, 0, -i\}$, and behave as $(\text{const})z^{2h_W-2}$ at infinity. Given an element $f \in \mathcal{F}(h_W)$ and a point $x \in \mathbb{C}$ we define the formal sum of modes of W

$$W[f, x] = \sum_{m \in \mathbb{Z}} a_{m+h_W-1} W_m, \quad \text{where} \quad f(x + \zeta) = \sum_{m \in \mathbb{Z}} a_m \zeta^m. \quad (\text{A.1})$$

With this definition $W[z^{m+h_W-1}, 0] = W_m$, and $z^{m+h_W-1} \in \mathcal{F}(h_W)$ for $m \leq h_W - 1$, i.e. precisely when $\langle 0|W_m = 0$. For the same reason, $\langle 0|W[f, 0] = 0$ for all $f \in \mathcal{F}(h_W)$. The invariance condition for the conformal three-point blocks β with insertions of M, N, \mathcal{V} at $i, -i, 0$, respectively, is then obtained by inserting the contour integral $\oint f(z)W(z) dz$ around infinity and deforming the contour. One obtains

$$\beta(W[f, i]p, q, v) + \beta(p, W[f, -i]q, v) + \beta(p, q, W[f, 0]v) = 0, \quad (\text{A.2})$$

where $W \in \mathcal{V}$ is quasiprimary, and $f \in \mathcal{F}(h_W)$, $p \in M, q \in N$ and $v \in \mathcal{V}$ are arbitrary.

This translates into the following definition for $B(\hat{\mathcal{H}})$: given a $\mathcal{V} \times \bar{\mathcal{V}}$ -representation $\hat{\mathcal{H}}$, the space $B(\hat{\mathcal{H}})$ consists of all bilinear maps $b : \hat{\mathcal{H}} \times \mathcal{V} \rightarrow \mathbb{C}$ with the property that for all $u \in \hat{\mathcal{H}}, v \in \mathcal{V}$, and for all quasi-primary $W \in \mathcal{V}$ and all $f \in \mathcal{F}(h_W)$,

$$b(W[f, i]u, v) + b(\bar{W}[f, -i]u, v) + b(u, W[f, 0]v) = 0. \quad (\text{A.3})$$

Here, by $\bar{W}[f, x]$ we mean the formal sum $\sum_{m \in \mathbb{Z}} a_{m+h_W-1} \bar{W}_m$, where the coefficients a_m are defined as in (A.1).

Appendix B. The kernel of b is a $\mathcal{V} \times \bar{\mathcal{V}}$ -representation

Given a $\mathcal{V} \times \bar{\mathcal{V}}$ -representation \hat{U} together with a pairing $b \in B(\hat{U})$ we will prove that the kernel of $b : \hat{U} \rightarrow \mathcal{V}^*$ is a $\mathcal{V} \times \bar{\mathcal{V}}$ -subrepresentation of \hat{U} . This statement is implied by the following lemma.

Lemma. *Let $W \in \mathcal{V}$ be quasi-primary and let $u \in \hat{U}$ be such that $b(u, v) = 0$ for all $v \in \mathcal{V}$. Then also $b(W_m u, v) = 0 = b(\bar{W}_m u, v)$ for all $v \in \mathcal{V}$ and $m \in \mathbb{Z}$.*

Proof. We will prove the assertion by induction on the mode number m . By definition of a $\mathcal{V} \times \bar{\mathcal{V}}$ -representation, for every vector u there is an integer $M(u)$ such that $W_m u = 0 = \bar{W}_m u$

for all $m \geq M(u)$. To start the induction we note that $b(W_m u, v) = 0 = b(\bar{W}_m u, v)$ for all $v \in \mathcal{V}$ and $m \geq M(u)$. Suppose now we have proved the statement for all $m > m_0$. Consider the function

$$f(z) = (z - i)^{m_0+h_w-1} ((z+i)/(2i))^{M(u)+h_w-1} (z/i)^{-m_0-M(u)} \in \mathcal{F}(h_w). \quad (\text{B.1})$$

Then $W[f, i] = W_{m_0} + (\text{higher})$, where (higher) stands for terms with mode number greater than m_0 . Furthermore we have $W[f, -i] = (\text{const}) \cdot W_{M(u)} + (\text{higher})$ so that $\bar{W}[f, -i]u = 0$. We compute

$$\begin{aligned} b(W_{m_0} u, v) &\stackrel{(1)}{=} b(W[f, i]u, v) - b((\text{higher})u, v) \stackrel{(2)}{=} b(W[f, i]u, v) \\ &\stackrel{(3)}{=} -b(\bar{W}[f, -i]u, v) - b(u, W[f, 0]v) \stackrel{(4)}{=} 0. \end{aligned} \quad (\text{B.2})$$

In step 2 we employed the induction assumption to set the second term to zero, step 3 uses the definition of $B(\bar{U})$, and in step 4 the first term vanishes because $\bar{W}[f, -i]u = 0$ and the second term is of the form $b(u, v')$ for some $v' \in \mathcal{V}$, which is zero by assumption. Similarly one can show that $b(\bar{W}_{m_0} u, v) = 0$ for all v . \square

Appendix C. Fusion rules for \mathcal{W}_p -representations

The product of irreducible representations in the Grothendieck ring has been conjectured in [29]. In this appendix, we extend this conjecture to the fusion product \otimes of the irreducible and projective representations.

The (additive) Grothendieck group $K_0(\mathcal{C})$ of an Abelian category \mathcal{C} is the quotient of the free Abelian group generated by isomorphism classes $[U]$ of objects in \mathcal{C} by the subgroup generated by the relations $[K] + [Q] = [M]$ for each short exact sequence $0 \rightarrow K \rightarrow M \rightarrow Q \rightarrow 0$. If \mathcal{C} is also monoidal and the tensor (fusion) functor \otimes is exact, then we obtain a ring structure on $K_0(\mathcal{C})$ via $[U] \cdot [V] = [U \otimes V]$.

Denote the category of \mathcal{W}_p -modules by \mathcal{C}_p . The Grothendieck group is freely generated by the $2p$ classes of the irreducible representations $[U_s^\pm]$, $s = 1, \dots, p$. From [29, section 2.4] we have the exact sequences

$$0 \rightarrow U_s^v \rightarrow M_{\varepsilon,s}^v \rightarrow U_{p-s}^{-v} \rightarrow 0, \quad 0 \rightarrow U_s^v \rightarrow N_s^v \rightarrow U_{p-s}^{-v} \oplus U_{p-s}^{-v} \rightarrow 0, \quad (\text{C.1})$$

where $s = 1, \dots, p-1$ and $v, \varepsilon = \pm$; in addition we have the first sequence of (4.3). These give the following identities in $K_0(\mathcal{C}_p)$:

$$[P_s^v] = 2[U_s^v] + 2[U_{p-s}^{-v}], \quad [N_s^v] = [U_s^v] + 2[U_{p-s}^{-v}], \quad [M_{\pm,s}^v] = [U_s^v] + [U_{p-s}^{-v}]. \quad (\text{C.2})$$

Since the exact sequences split when considered as sequences of graded vector spaces (and not as sequences of \mathcal{W}_p -modules) the identities (C.2) are sum rules for the characters of the corresponding representations.

Since $M_{\pm,s}^v$ are submodules of N_s^v , the exact sequence $0 \rightarrow M_{\pm,s}^v \rightarrow N_s^v \rightarrow X \rightarrow 0$ implies the relation $[N_s^v] = [M_{\pm,s}^v] + [X]$, which together with (C.2) shows $X \cong U_{p-s}^{-v}$. A similar argument shows that in $0 \rightarrow M_{\varepsilon,s}^v \rightarrow P_s^v \rightarrow Y \rightarrow 0$ we either have $Y \cong M_{\pm\varepsilon,p-s}^{-v}$ (since the quotient $N_s^v/M_{\varepsilon,s}^v \cong U_{p-s}^{-v}$ is embedded in $P_s^v/M_{\varepsilon,s}^v \cong Y$) or $Y \cong U_s^v \oplus U_{p-s}^{-v}$. The second possibility is excluded since P_s^v is already the projective cover of U_s^v . The choice of sign in $Y \cong M_{\pm\varepsilon,p-s}^{-v}$ is a convention which can be reversed by redefining $M_{\varepsilon,s}^{\pm, \text{new}} = M_{\pm\varepsilon,s}^{\pm, \text{old}}$. We fix the convention as stated in (4.3).

Assuming that the tensor (fusion) functor on \mathcal{C}_p is exact, in [29] the following conjecture for the ring structure on $K_0(\mathcal{C}_p)$ is made (we follow the exposition in [32, section 6.3]). The product is commutative, and ordering the factors such that $1 \leq t \leq s \leq p$ we have

$$\begin{aligned}
 [U_s^\mu] \cdot [U_t^\nu] &= \sum_{i=1}^t [\hat{U}_{s-t+2i-1}^{\mu\nu}], \\
 [\hat{U}_x^\pm] &= \begin{cases} [U_x^\pm] & \text{for } 1 \leq x \leq p, \\ [U_{2p-x}^\pm] + 2[U_{x-p}^\mp] & \text{for } p+1 \leq x \leq 2p-1. \end{cases} \quad (\text{C.3})
 \end{aligned}$$

The Grothendieck ring does not determine the fusion product of representations uniquely. However, we can arrive at a convincing proposal for the fusion product of irreducible and projective representations using the analysis of the fusion of Virasoro (rather than \mathcal{W}_p) representations in [26]. This leads to the following natural ansatz for the fusion product of two irreducible \mathcal{W}_p -representations:

$$\begin{aligned}
 U_s^\mu \otimes U_t^\nu &= \bigoplus_{r=|s-t|+1;2}^{\min(s+t-1, 2p-s-t-1)} U_r^{\mu\nu} \oplus \bigoplus_{r=2p-s-t+1;2}^M P_r^{\mu\nu} \quad \text{where} \\
 M &= \begin{cases} p-1 & \text{if } p+s+t \text{ even,} \\ p & \text{if } p+s+t \text{ odd.} \end{cases} \quad (\text{C.4})
 \end{aligned}$$

The ‘; 2’ means the above direct sums are taken in steps of 2. On the level of the Grothendieck ring (C.4) is equivalent to (C.3). Note also that U_1^- is a simple current, $U_1^- \otimes U_s^\varepsilon = U_s^{-\varepsilon}$.

According to proposition 2.2 in [58], tensor (fusion) products involving at least one projective module are already fixed by the Grothendieck ring (to apply this result we need to assume that \mathcal{C}_p is a finite tensor category, see [58] for details). The proposition states that

$$P_i \otimes Z = \bigoplus_{j,k \in \text{Irr}} N_{kj}^i [Z : U_j] P_k, \quad (\text{C.5})$$

where, for $j \in \text{Irr}$, U_j is the simple object with label j , P_j its projective cover, and Z an arbitrary object in \mathcal{C}_p . The N_{kj}^i are the structure constants of the Grothendieck ring (C.3), $[U_k] \cdot [U_j] = \sum_{i \in \text{Irr}} N_{kj}^i [U_i]$ and $[Z : U_j] \in \mathbb{Z}_{\geq 0}$ gives the decomposition of Z in $K_0(\mathcal{C})$ as $[Z] = \sum_{j \in \text{Irr}} [Z : U_j] [U_j]$.

In writing (C.5) we have assumed that the simple objects are self-dual, i.e. that for the \mathcal{W}_p -representations we have $U_s^{\pm*} \cong U_s^\pm$. (The statement without this assumption can be found in [58, prop. 2.2].) We will also assume that $P_s^{\pm*} \cong P_s^\pm$.

Equations (C.4) and (C.5) determine now all fusion products $U_s^\mu \otimes U_t^\nu$, $U_s^\mu \otimes P_t^\nu$ and $P_s^\mu \otimes P_t^\nu$ uniquely. Explicitly, we find that

$$U_s^\mu \otimes P_t^\nu = \bigoplus_{r=|s-t|+1;2}^{\min(s+t-1, 2p-s-t-1)} P_r^{\mu\nu} \oplus \bigoplus_{r=2p-s-t+1;2}^M 2P_r^{\mu\nu} \oplus \bigoplus_{r=p+1+t-s;2}^{\hat{M}} 2P_r^{-\mu\nu} \quad \text{for } t \leq p-1, \quad (\text{C.6})$$

where M is defined as in (C.4) and

$$\hat{M} = \begin{cases} p-1 & \text{if } s+t \text{ even} \\ p & \text{if } s+t \text{ odd.} \end{cases} \quad (\text{C.7})$$

Finally,

$$P_s^\mu \otimes P_t^\nu = 2U_s^\mu \otimes P_t^\nu \oplus 2U_{p-s}^{-\mu} \otimes P_t^\nu \quad \text{for } s, t \leq p-1, \quad (\text{C.8})$$

where the right-hand side is defined by (C.6). It is straightforward to check that the fusion product defined by (C.4), (C.6) and (C.8) is compatible with the product (C.3) of

the Grothendieck ring. We have also tested for a large number of values for p that the fusion product is associative (as it must be).

Let us also compare our proposal for the fusion products with the results in [65, section 6], which are based on generalized versions of the Verlinde formula. Formula (C.4) agrees with [65, equation (6.16)] if we identify $U_s^+ = [h_{1,s}]$, $U_s^- = [h_{1,3p-s}]$ and $P_s^+ = [\tilde{h}_{1,2p-s}]$. However, the method used in [65] does not distinguish between P_s^+ and P_{p-s}^- . Keeping this in mind, one can recover [65, equation (6.19)] by starting from (C.6) and replacing in addition $P_s^- \mapsto [\tilde{h}_{1,p+s}]$.

To find the dimension of the intertwiner spaces $\text{Hom}_V(P_s^\mu, P_t^\nu)$ one can proceed as follows. First note that by the properties of duals and by uniqueness of the projective cover we have

$$\text{Hom}_V(P_s^\mu, P_t^\nu) \cong \text{Hom}_V(P_s^\mu \otimes P_t^{\nu*}, U_0^+), \quad \dim \text{Hom}_V(P_s^\mu, U_0^+) = \delta_{\mu,+} \delta_{s,0}. \quad (\text{C.9})$$

As mentioned above we assume that $P_s^{\pm*} \cong P_s^\pm$. To obtain the dimension of $\text{Hom}_V(P_s^\mu, P_t^\nu)$ it is thus sufficient to compute the multiplicity of P_0^+ in $P_s^\mu \otimes P_t^\nu$. From (C.5) we find

$$P_s^\mu \otimes P_t^\nu \cong [P_t^\nu : U_s^\mu] P_0^+ \oplus (\text{other projectives}). \quad (\text{C.10})$$

The multiplicity in $K_0(\mathcal{C}_p)$ follows from (C.2) to be $[P_t^\nu : U_s^\mu] = 2\delta_{\nu,\mu} \delta_{s,t} + 2\delta_{\nu,-\mu} \delta_{s,p-t}$. This shows that the dimension of the intertwiner spaces is indeed as proposed in (4.6).

Appendix D. The structure of $\hat{\mathcal{H}}_s/\mathcal{N}_s$

In this appendix we want to prove (4.26). To do so we recall that N_s^ε is a subrepresentation of P_s^ε . In each generalized eigenspace $(P_s^\varepsilon)_h$ of P_s^ε of eigenvalue h choose a sub-vector space $(V_s^\varepsilon)_h$ such that $(P_s^\varepsilon)_h = (V_s^\varepsilon)_h \oplus (N_s^\varepsilon)_h$. In words, $(V_s^\varepsilon)_h$ and $(N_s^\varepsilon)_h$ have intersection $\{0\}$ and together span $(P_s^\varepsilon)_h$. We will write $V_s^\varepsilon = \bigoplus_{h \in \mathbb{R}} (V_s^\varepsilon)_h$. The vector space V_s^ε is not a \mathcal{W}_p -subrepresentation of P_s^ε .

Consider the projectors $\Pi_s^\varepsilon : P_s^\varepsilon \rightarrow P_s^\varepsilon$ which act as the identity on N_s^ε , and as zero on V_s^ε (these are not intertwiners of the \mathcal{W}_p -action). Using the Π_s^ε we can define a projector $\Pi_s : \hat{\mathcal{H}}_s \rightarrow \hat{\mathcal{H}}_s$ by $\Pi_s = (\Pi_s^+ \otimes \text{id}) \oplus (\Pi_{p-s}^- \otimes \text{id})$. By construction we have

$$\ker(\Pi_s) = (V_s^+ \otimes \bar{P}_s^{+*}) \oplus (V_{p-s}^- \otimes \bar{P}_{p-s}^{-*}). \quad (\text{D.1})$$

It is proved in the following subsection that the restriction of Π_s to \mathcal{N}_s is injective. This implies that $\mathcal{N}_s \cap \ker(\Pi_s) = \{0\}$. Consider the quotient $\hat{\mathcal{H}}_s/\mathcal{N}_s$, and for an element $x \in \hat{\mathcal{H}}_s$ denote the class in $\hat{\mathcal{H}}_s/\mathcal{N}_s$ by $[x]$. We will show that every element of $\hat{\mathcal{H}}_s/\mathcal{N}_s$ can be written as $[k]$ with $k \in \ker(\Pi_s)$. This then implies that \mathcal{N}_s and $\ker(\Pi_s)$ together span $\hat{\mathcal{H}}_s$.

It is enough to consider elements of $\hat{\mathcal{H}}_s$ of the form $(v + \eta) \otimes \bar{q}$ where either $v \in V_s^+$, $\eta \in N_s^+$, $\bar{q} \in \bar{P}_s^{+*}$, or $v \in V_{p-s}^-$, $\eta \in N_{p-s}^-$, $\bar{q} \in \bar{P}_{p-s}^{-*}$. Take the first case, for concreteness. The map $e_v : P_{p-s}^- \rightarrow P_s^+$ has image $M_{v,s}^+$ and kernel $M_{v,p-s}^-$. This implies that e_v maps $N_{p-s}^- \subset P_{p-s}^-$ to U_s^+ and that we can write an arbitrary element $m_v \in M_{v,s}^+ \subset P_s^+$ as $m_v = e_v(w) + u$ for appropriate $w \in V_{p-s}^-$, $u \in U_s^+$. The element u in turn can be expressed as $u = e_-(e_+(w'))$ for some $w' \in V_s^+$. Altogether we see that for any $\eta \in N_s^+$

$$\exists w_+, w_- \in V_{p-s}^-, \quad w_0 \in V_s^+ : \eta = e_+(w_+) + e_-(w_-) + e_-(e_+(w_0)). \quad (\text{D.2})$$

Since the images of $\text{id} \otimes \bar{e}_v^* - e_v \otimes \text{id}$ are in \mathcal{N}_s , it follows that in the quotient space $\hat{\mathcal{H}}_s/\mathcal{N}_s$ $[\eta \otimes \bar{q}] = [w_+ \otimes \bar{e}_+^*(\bar{q})] + [w_- \otimes \bar{e}_-^*(\bar{q})] + [w_0 \otimes \bar{e}_+^*((\bar{e}_-^*(\bar{q})))]$. Thus $[(v + \eta) \otimes \bar{q}]$ can be written as a sum of four terms all of which lie in $\ker(\Pi_s)$. Thus we have shown that

$$\hat{\mathcal{H}}_s = (V_s^+ \otimes \bar{P}_s^{+*}) \oplus (V_{p-s}^- \otimes \bar{P}_{p-s}^{-*}) \oplus \mathcal{N}_s \quad (\text{D.3})$$

as a vector space with generalized (L_0, \bar{L}_0) -grading. Since Π_s commutes with the action of $\bar{\mathcal{W}}_p$ and since \mathcal{N}_s is a $\mathcal{W}_p \times \bar{\mathcal{W}}_p$ -subrepresentation, the decomposition (D.3) is also preserved by the $\bar{\mathcal{W}}_p$ -action (but not by the \mathcal{W}_p -action).

Finally, we need to show that V_s^ε is isomorphic (as a graded vector space) to U_s^ε . To see this we observe that the surjection $\pi_s^\varepsilon : P_s^\varepsilon \twoheadrightarrow U_s^\varepsilon$ in the first exact sequence in (4.3) has kernel N_s^ε . In particular, π_s^ε restricts to a bijection $V_s^\varepsilon \rightarrow U_s^\varepsilon$ which is compatible with the generalized L_0 -grading (but not with the action of the \mathcal{W}_p -modes, or even with the action of L_0 itself). This then proves (4.26).

D.1. The projection Π_s is injective on \mathcal{N}_s

Recall the decomposition $(P_s^\varepsilon)_h = (V_s^\varepsilon)_h \oplus (N_s^\varepsilon)_h$ chosen above, and the inclusions (4.2). Let us also choose subspaces $(U_s^\varepsilon)_h$ and $(S_{\pm,s}^\varepsilon)_h$ of $(N_s^\varepsilon)_h$ such that $(U_s^\varepsilon)_h$ is the generalized L_0 -eigenspace of eigenvalue h of $U_s^\varepsilon \subset P_s^\varepsilon$, as well as $(M_{\pm,s}^\varepsilon)_h = (S_{\pm,s}^\varepsilon)_h \oplus (U_s^\varepsilon)_h$. As was the case for V_s^ε , the subspaces $S_{\pm,s}^\varepsilon = \bigoplus_{h \in \mathbb{R}} (S_{\pm,s}^\varepsilon)_h$ are not \mathcal{W}_p -submodules. We have now chosen the direct sum decompositions

$$P_s^\varepsilon = V_s^\varepsilon \oplus S_{+,s}^\varepsilon \oplus S_{-,s}^\varepsilon \oplus U_s^\varepsilon, \quad N_s^\varepsilon = S_{+,s}^\varepsilon \oplus S_{-,s}^\varepsilon \oplus U_s^\varepsilon, \quad M_{\pm,s}^\varepsilon = S_{\pm,s}^\varepsilon \oplus U_s^\varepsilon. \quad (\text{D.4})$$

According to the construction in section 4.3 every element k of \mathcal{N}_s can be written as a sum of the form

$$k = \sum_{\nu, \alpha} (p_\alpha^\nu \otimes \bar{e}_\nu^*(\bar{q}_\alpha^\nu) - e_\nu(p_\alpha^\nu) \otimes \bar{q}_\alpha^\nu) + \sum_{\nu, \beta} (x_\beta^\nu \otimes \bar{e}_\nu^*(\bar{y}_\beta^\nu) - e_\nu(x_\beta^\nu) \otimes \bar{y}_\beta^\nu), \quad (\text{D.5})$$

where $p_\alpha^\nu \in P_s^+$, $\bar{q}_\alpha^\nu \in \bar{P}_{p-s}^{-*}$ and $x_\beta^\nu \in P_{p-s}^-$, $\bar{y}_\beta^\nu \in \bar{P}_s^{+*}$, for $\nu = \pm$. We have to show that for any $k \in \mathcal{N}_s$

$$\Pi_s(k) = 0 \quad \Rightarrow \quad k = 0, \quad (\text{D.6})$$

where Π_s is the projector defined just before (D.1). Since the image of e_ν lies in N_s^ε , we have $\Pi_s \circ e_\nu = e_\nu$. Thus

$$\Pi_s(k) = \sum_{\nu, \alpha} (\Pi_s(p_\alpha^\nu) \otimes \bar{e}_\nu^*(\bar{q}_\alpha^\nu) - e_\nu(p_\alpha^\nu) \otimes \bar{q}_\alpha^\nu) + \sum_{\nu, \beta} (\Pi_s(x_\beta^\nu) \otimes \bar{e}_\nu^*(\bar{y}_\beta^\nu) - e_\nu(x_\beta^\nu) \otimes \bar{y}_\beta^\nu). \quad (\text{D.7})$$

The summands in the sums over α and β lie in different direct summands of $\hat{\mathcal{H}}_s$, and so the equation $\Pi_s(k) = 0$ implies that both sums in (D.7) have to vanish separately. Consider the first sum. We will prove below that

$$\sum_{\nu, \alpha} \Pi_s(p_\alpha^\nu) \otimes \bar{e}_\nu^*(\bar{q}_\alpha^\nu) - e_\nu(p_\alpha^\nu) \otimes \bar{q}_\alpha^\nu = 0 \quad \Rightarrow \quad \sum_{\nu, \alpha} p_\alpha^\nu \otimes \bar{e}_\nu^*(\bar{q}_\alpha^\nu) - e_\nu(p_\alpha^\nu) \otimes \bar{q}_\alpha^\nu = 0. \quad (\text{D.8})$$

The corresponding statement for the second sum in (D.7) can be seen analogously, and the two statements together imply (D.6), i.e. that Π_s is injective on \mathcal{N}_s .

According to the decomposition (D.4) the vectors $p_\alpha^\nu \in P_s^+$ can be written as

$$p_\alpha^\nu = v_\alpha^\nu + m_{\alpha,+}^\nu + m_{\alpha,-}^\nu + u_\alpha^\nu, \quad \text{where} \quad v_\alpha^\nu \in V_s^+, \quad m_{\alpha,\pm}^\nu \in S_{\pm,s}^+, \quad u_\alpha^\nu \in U_s^+. \quad (\text{D.9})$$

Furthermore we have the induced decomposition of the dual spaces $P_s^{\varepsilon*} = V_s^{\varepsilon*} \oplus S_{+,s}^{\varepsilon*} \oplus S_{-,s}^{\varepsilon*} \oplus U_s^{\varepsilon*}$ and we will write \bar{q}_α^ν as

$$\bar{q}_\alpha^\nu = \bar{v}_\alpha^\nu + \bar{m}_{\alpha,+}^\nu + \bar{m}_{\alpha,-}^\nu + \bar{u}_\alpha^\nu, \quad \text{where} \quad \bar{v}_\alpha^\nu \in \bar{V}_{p-s}^{-*}, \quad \bar{m}_{\alpha,\pm}^\nu \in \bar{S}_{\pm,p-s}^{-*}, \quad \bar{u}_\alpha^\nu \in \bar{U}_{p-s}^{-*}. \quad (\text{D.10})$$

Here it is understood that, despite the similarity in notation, v_α^v and \bar{v}_α^v are independent, and similar for the other vectors in (D.9) and (D.10). Now for e_v acting on P_s^+ we have

$$\ker(e_v) = S_{v,s}^+ \oplus U_s^+, \quad \ker(e_- \circ e_+) = S_{+,s}^+ \oplus S_{-,s}^+ \oplus U_s^+, \quad (\text{D.11})$$

and e_v acts injectively on $V_s^+ \oplus S_{-v,s}^+$, while $n = e_- \circ e_+$ is injective on V_s^+ . Dually, for \bar{e}_v^* acting on \bar{P}_{p-s}^* we have

$$\ker(\bar{e}_v^*) = \bar{V}_{p-s}^* \oplus \bar{S}_{-v,p-s}^*, \quad \ker(\bar{e}_+^* \circ \bar{e}_-^*) = \bar{V}_{p-s}^* \oplus \bar{S}_{+,p-s}^* \oplus \bar{S}_{-,p-s}^*, \quad (\text{D.12})$$

\bar{e}_v^* is injective on $\bar{S}_{v,p-s}^* \oplus \bar{U}_{p-s}^*$ and $\bar{n}^* = \bar{e}_+^* \circ \bar{e}_-^*$ is injective on \bar{U}_{p-s}^* . Using these decompositions and kernels, we can write the condition of the implication (D.8) as

$$\sum_{v,\alpha} ((m_{\alpha,+}^v + m_{\alpha,-}^v + u_\alpha^v) \otimes \bar{e}_v^*(\bar{m}_{\alpha,v}^v + \bar{u}_\alpha^v) - e_v(v_\alpha^v + m_{\alpha,-v}^v) \otimes (\bar{v}_\alpha^v + \bar{m}_{\alpha,+}^v + \bar{m}_{\alpha,-}^v + \bar{u}_\alpha^v)) = 0. \quad (\text{D.13})$$

When applying $e_{-\mu} \otimes \bar{n}^*$ to this equation only the second term of the sum survives and we obtain, for $\mu = \pm$,

$$\sum_{\alpha} e_{-\mu}(e_\mu(v_\alpha^\mu)) \otimes \bar{n}^*(\bar{u}_\alpha^\mu) = 0 \quad \Rightarrow \quad \sum_{\alpha} v_\alpha^\mu \otimes \bar{u}_\alpha^\mu = 0, \quad (\text{D.14})$$

where in the implication we used that n is injective on V_s^+ and \bar{n}^* is injective on \bar{U}_{p-s}^* . Applying $\text{id} \otimes \bar{n}^*$ to (D.13) gives

$$\sum_{v,\alpha} e_v(v_\alpha^v + m_{\alpha,-v}^v) \otimes \bar{n}^*(\bar{u}_\alpha^v) = 0 \quad \Rightarrow \quad \sum_{v,\alpha} e_v(m_{\alpha,-v}^v) \otimes \bar{n}^*(\bar{u}_\alpha^v) = 0, \quad (\text{D.15})$$

where the implication follows from the result (D.14). Finally, applying $e_{-\mu} \otimes \bar{e}_\mu^*$ to (D.13) results in

$$\sum_{\alpha} (e_{-\mu}(m_{\alpha,\mu}^{-\mu}) \otimes \bar{n}^*(\bar{u}_\alpha^{-\mu}) - n(v_\alpha^\mu) \otimes \bar{e}_\mu^*(\bar{m}_{\alpha,\mu}^\mu + \bar{u}_\alpha^\mu)) = 0. \quad (\text{D.16})$$

Summing this equation over $\mu = \pm$ and using (D.15) removes the first term, so that we are left with

$$\sum_{\alpha,\mu} n(v_\alpha^\mu) \otimes \bar{e}_\mu^*(\bar{m}_{\alpha,\mu}^\mu + \bar{u}_\alpha^\mu) = 0 \quad \Rightarrow \quad \sum_{\alpha,\mu} v_\alpha^\mu \otimes \bar{e}_\mu^*(\bar{m}_{\alpha,\mu}^\mu + \bar{u}_\alpha^\mu) = 0, \quad (\text{D.17})$$

where for the implication one uses that n is injective on V_s^+ . Adding (D.17) to (D.13) gives precisely the result of the implication (D.8), thus completing the proof of (D.8).

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